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TWO-WEIGHTED ESTIMATES FOR SOME SUBLINEAR OPERATORS ON GENERALIZED WEIGHTED MORREY SPACES AND APPLICATIONS

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Abstract
In this paper, we investigate the boundedness of sublinear operators generated by fractional integrals as well as sublinear operators generated by Calderón-Zygmund operators on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces. In particular, we are interested in the strong-type estimate for $1 < p < \infty$ and the weak estimate for $p = 1$. Under some assumptions, we prove that the operators and their commutator with a BMO function are bounded on those function spaces with different weights. The results then imply the boundedness of fractional integrals with Gaussian kernel bounds as well as with rough kernels, fractional maximal integrals with rough kernels, and sublinear operators with rough kernels generated by Calderón-Zygmund operators. Using the results, we obtain some regularity properties of the solution of some partial differential equations.

Keywords Sublinear operators · Fractional integrals · Calderón-Zygmund operator · Commutators · $\boldsymbol{A_{p,q}}$ weights · Generalized weighted Morrey spaces

Introduction

We shall discuss the boundedness of sublinear operators on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces with different weights and then apply the results to the regularity properties of the solution of some partial differential equations.

The classical Morrey spaces $L^{p,\lambda}$ were first introduced by C. B. Morrey in 1938 [40]. After the WW II, a large number of investigations regarding the function spaces have been carried out. The reader may find some recent works in [3, 26, 33, 38, 47] and the references therein.

Throughout this paper, we denote by $B(a, r)$ an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$. For a set E in $\mathbb{R}^n$, we denote by E^{c} the complement of E. Moreover, if E is a measurable set in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of E.

Let $0 \leq \alpha < n$, $1 \leq p < n/\alpha$, and $1/q = 1/p - \alpha/n$. We consider a sublinear operator S_α such that for every $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{1/s} < \infty, \quad (a, r) \in \mathbb{R}^n \times \mathbb{R}^+$$

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where $1 < s < p$ for $1 < p < n/\alpha$ or $s = 1$ for $p = 1$, the following inequality

$$\sup_{x\in B(a,r)} \left|S_\alpha\left(f\cdot\chi_{B(a,2r)^c}\right)(x)\right| \le C\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{1/s} \tag{1.1}$$

holds. For $\alpha \in (0, n)$, this operator was introduced by Y. Wang [57].

In addition, for $b \in BMO$, the commutator of the sublinear operator S_α where $0 \le \alpha < n$ is defined by

$$[b, S_\alpha](f)(x) = S_\alpha((b(x) - b(\cdot))f)(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

whenever this makes sense. In general, to make sure that $[b, S_\alpha](f)$ is defined, we require that this expression is finite almost everywhere. In particular, if we split f into the local and the global parts, the right-hand side of Eq. 1.1 must be finite for both parts, in place of f.

In this paper, we are not interested in particular formulae for S_α. However, for $\alpha = 0$, the sublinear operator S_0 covers the sublinear operator S generated by Calderón-Zygmund operator satisfying

$$|S(f)(x)| \le C\int_{\mathbb{R}^n}\frac{|f(y)|}{|x-y|^n}dy \tag{1.3}$$

for $f \in L^1$ with compact support and $x \notin \operatorname{supp}(f)$. Observe that this operator satisfies

$$\sup_{x\in B(a,r)} |S(f\cdot\chi_{B(a,2r)^c})(x)| \le C\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}}\left(\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{\frac{1}{s}} \tag{1.4}$$

which is the inequality Eq. 1.1 for S_0. In other words, the inequality Eq. 1.4 is weaker than Eq. 1.3. Hereafter, without confusion, we will write $S := S_0$.

For $0 < \alpha < n$, the fractional integral operator or the Riesz potential I_α also satisfies (1.1). Thus, our S_α covers S and I_α.

There are many other operators (and their commutators) satisfying the assumption (1.1) as well as one formulated in Eq. 1.5 below. For $0 < \alpha < n$, besides the Riesz potential, one may consider fractional integrals related to operators with Gaussian kernel bounds, fractional operators with rough kernels, and fractional maximal operators with rough kernels. Meanwhile, for $\alpha = 0$, the Hardy–Littlewood maximal operator, generalized Calderón–Zygmund operators, Carleson type maximal operators, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci–Stein's oscillatory singular integrals, and Bochner–Riesz means satisfy Eqs. 1.1 and 1.5.

Assuming that these sublinear operators and their commutators are bounded on weighted Lebesgue spaces (or weighted weak Lebesgue spaces), Wang obtained the following theorem on their boundedness property on generalized weighted Morrey spaces (or generalized weighted weak Morrey spaces) for $0 < \alpha < n$.

Theorem 1.1 *[57] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w^s \in A_{p/s,q/s}$ where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$, and S_α be a sublinear operator satisfying (Eq. refsublinear for any $(a, r) \in \mathbb{R}^n \in \mathbb{R}^+$.*

1. *Suppose that (ψ_1, ψ_2) satisfies the condition*

$$\int_r^{\infty}\frac{\operatorname{ess\,inf}_{t<l<\infty}\psi_1(a,l)w^p(B(a,l))^{\frac{1}{p}}}{w^p(B(a,t))^{\frac{1}{p}}}\frac{dt}{t} \le C\psi_2(a,r), \quad (a,r)\in\mathbb{R}^n\times\mathbb{R}^+.$$

If S_α is bounded from L^{p,w^p} to L^{q,w^q} for $1 < p < n/\alpha$ and from $L^{1,w}$ to $WL^{1,w}$, then S_α is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$ for $1 < p < n/\alpha$ and from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$.

2. *Suppose that (ψ_1, ψ_2) satisfies the condition*

$$\int_r^{\infty}\left(1+\ln\frac{r}{t}\right)\frac{\operatorname{ess\,inf}_{t<l<\infty}\psi_1(a,l)w^p(B(a,l))^{\frac{1}{p}}}{w^p(B(a,t))^{\frac{1}{p}}}\frac{dt}{t} \le C\psi_2(a,r), \quad (a,r)\in\mathbb{R}^n\times\mathbb{R}^+.$$

If $b \in BMO$ and $[b, S_\alpha]$ is bounded from L^{p,w^p} to L^{q,w^q} for $1 < p < n/\alpha$ and from $L^{1,w}$ to $WL^{1,w}$, then $[b, S_\alpha]$ is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$ for $1 < p < n/\alpha$ and from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$.

In this paper, we extend Wang's results. We investigate the boundedness of the sublinear operators on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces with different weights (which we shall define in the "Preliminaries: Definitions and Lemmas" section), where $w^s \in A_{p/s,q/s}$ is replaced by $(w_1^s, w_2^s) \in A_{p/s,q/s}$ for $1 \le p < n/\alpha$. Moreover, we also investigate the operator S_0 for the case $\alpha = 0$ to extend the known results for the sublinear operators generated by Calderón-Zygmund operators on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces.

Related to the generalized weighted mixed-Morrey spaces, we modify the sublinear S_α satisfying Eqs. 1.1 or 1.4 to become T_α as a sublinear operator with the conditions

$$\sup_{x\in B(a,r)} \left|T_\alpha\left(f\cdot\chi_{B(a,2r)^c}\right)(x,t)\right| \le C\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|f(y,t)|^s dy\right)^{1/s}, \tag{1.5}$$

for any $(a,r)\in\mathbb{R}^n\times\mathbb{R}^+$ and $t\in(0,T)$, where $1<s<p$ for $1<p<n/\alpha$ and $s=1$ for $p=1$, and $T>0$ is fixed. The commutator is then given by

$$[b,T_\alpha](f)(x,t) = T_\alpha((b(x)-b(\cdot))f)(x,t), \quad x\in\mathbb{R}^n, t\in(0,T).$$

We shall assume that S_α, $[b,S_\alpha]$, T_α, and $[b,T_\alpha]$ are bounded from L^{p,w_1^p} to L^{q,w_2^q} (or other relevant space) where $1/q = 1/p-\alpha/n$ and S_α, $[b,S_\alpha]$, T_α, and $[b,T_\alpha]$ are bounded from L^{p,w_1} to L^{p,w_2} (or other relevant space) for $\alpha=0$. Although these initial assumptions only guarantee that the operators can act on functions in weighted Lebesgue spaces, there is no a priori guarantee that they can act on functions in generalized weighted Morrey spaces. However, as we shall see later, the boundedness of S_α, $[b,S_\alpha]$, T_α, and $[b,T_\alpha]$ from L^{p,w_1} to L^{p,w_2} will imply the boundedness of the operators and their commutators on the spaces of our interests.

To address the domain issue rigorously, we note that the series on the right-hand sides of Eqs. 1.1 and 1.5 are indeed finite when f belongs to the generalized weighted Morrey spaces or their extensions considered in our theorems. We shall verify this fact explicitly. Consequently, under appropriate conditions on the pairs (w_1,w_2) and (ψ_1,ψ_2), we can extend the domain of the sublinear operators to these broader spaces.

This paper is quite long. For convenience, we state our main results below. The definitions and the properties of A_p and $A_{p,q}$ weights will be presented in the "Preliminaries: Definitions and Lemmas" section, along with the definitions of the function spaces that we are working on. We also provide the definitions of the sublinear operators acting on the relevant function spaces in the "Preliminaries: Definitions and Lemmas" section. The proofs of the theorems will be presented in the "Proof of Theorem 1.2–1.5" and "Proof of Theorem 1.6–1.8 and Theorem 1.10–1.12" sections. Finally, some applications of our main results will be given in the "Applications to other operators" and "Applications to partial differential equations" sections.

The following are our main results in this manuscript. We begin our analysis by focusing on the fractional integral operator S_α (where $0<\alpha<n$) acting on generalized weighted Morrey spaces. The following theorem establishes the foundational result, showing that the boundedness of S_α on Lebesgue spaces can be lifted to generalized Morrey spaces, provided the weight functions satisfy a specific integral condition.

Theorem 1.2 *Let $0<\alpha<n$, $1\le p<n/\alpha$, $1/q=1/p-\alpha/n$, $(w_1^s,w_2^s)\in A_{p/s,q/s}$ and $w_2^s\in A_{p/s,q/s}$ where $1\le s<p$ for $1<p<n/\alpha$ and $s=1$ for $p=1$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n\times\mathbb{R}^+$ satisfy*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\int_r^\infty \frac{\inf_{t\le l<\infty}\psi_1(a,l)w_1^p(B(a,l))^{\frac{1}{p}}}{w_2^q(B(a,t))^{\frac{1}{q}}}\frac{dt}{t}<\infty. \tag{1.6}$$

If S_α is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1<p<n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then S_α is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$ for $1<p<n/\alpha$ and from $\mathcal{M}_{\psi_1}^{1,w_1}$ to $W\mathcal{M}_{\psi_2}^{q,w_2^q}$.

Next, we extend this result to the commutator $[b,S_\alpha]$ with a symbol $b\in BMO$. It is worth noting that the integral condition for the weights now incorporates an additional logarithmic factor, which is a standard characteristic when dealing with commutator estimates.

Theorem 1.3 *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $(w_1^s,w_2^s)\in A_{p/s,q/s}$ and $w_2^s\in A_{p/s,q/s}$ where $1\le s<p$, and $w_1\in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n\times\mathbb{R}^+$ satisfy*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)\frac{\inf_{t\le l<\infty}\psi_1(a,l)w_1^p(B(a,l))^{\frac{1}{p}}}{w_2^q(B(a,t))^{\frac{1}{q}}}\frac{dt}{t}<\infty. \tag{1.7}$$

If $b\in BMO$ and $[b,S_\alpha]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q}, then $[b,S_\alpha]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$. Moreover, $\|[b,S_\alpha]f\|_{\mathcal{M}_{\psi_1}^{q,w_2^q}}\le C\|b\|_*\|f\|_{\mathcal{M}_{\psi_2}^{p,w_1^p}}$ for $f\in\mathcal{M}_{\psi_1}^{p,w^p}$.

The previous theorems can be generalized into the framework of mixed-Morrey spaces (often involving space-time variables, which are useful in evolution PDEs). The following theorem translates the boundedness of the fractional operator T_α into this mixed setting.

Theorem 1.4 *Let ψ be a positive function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p_0 < \infty$, $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$ and $w_2^s \in A_{p/s,q/s}$ where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.6. If T_α is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then T_α is bounded from $\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$ for $1 < p < n/\alpha$ and from $\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{1,w_1})$ to $\mathcal{M}_\psi^{p_0,w}(0,T,W\mathcal{M}_{\psi_2}^{q,w_2^q})$.*

Analogously, we can establish the boundedness for the commutator $[b, T_\alpha]$ within the mixed-Morrey spaces.

Theorem 1.5 *Let ψ be a positive function on $\mathbb{R}^n \times (0,\infty)$, $1 \le p_0 < \infty$, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$ and $w_2^s \in A_{p/s,q/s}$ where $1 \le s < p$, and $w_1 \in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.7. If $b \in BMO$ and $[b, T_\alpha]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q}, then $[b, T_\alpha]$ is bounded from $\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$. Moreover, $\|[b,T_\alpha]f\|_{\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})} \le C\|b\|_*\|f\|_{\mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})}$ for $f \in \mathcal{M}_\psi^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$.*

We now shift our attention to the limiting case where $\alpha = 0$, which corresponds to singular integral operators (such as Calderón-Zygmund operators), denoted here by $S = S_0$. In this scenario, there is no fractional shift, meaning the index p is mapped back to p.

Theorem 1.6 *Let $1 \le p < \infty$, $(w_1, w_2) \in A_{p/s}$ and $w_2 \in A_{p/s}$ where $1 \le s < p$ for $1 < p < \infty$ and $s = 1$ for $p = 1$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\inf_{t\le l<\infty} \psi_1(a,l) w_1(B(a,l))^{\frac{1}{p}}}{w_2(B(a,t))^{\frac{1}{p}}} \frac{dt}{t} < \infty. \tag{1.8}$$

If S is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$ and from L^{1,w_1} to WL^{1,w_2}, then S is bounded from $\mathcal{M}_{\psi_1}^{p,w_1}$ to $\mathcal{M}_{\psi_2}^{p,w_2}$ for $1 < p < \infty$ and from $\mathcal{M}_{\psi_1}^{1,w_1}$ to $W\mathcal{M}_{\psi_2}^{1,w_2}$.

Following the established pattern, we present the result for the commutator $[b, S]$ in the singular case.

Theorem 1.7 *Let $1 < p < \infty$, $(w_1, w_2) \in A_{p/s}$ and $w_2 \in A_{p/s}$ where $1 \le s < p$, and $w_1 \in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\inf_{t\le l<\infty} \psi_1(a,l) w_1(B(a,l))^{\frac{1}{p}}}{w_2(B(a,l))^{\frac{1}{p}}} \frac{dt}{t} < \infty. \tag{1.9}$$

If $b \in BMO$ and $[b, S]$ is bounded from L^{p,w_1} to L^{p,w_2}, then $[b, S]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1}$ to $\mathcal{M}_{\psi_2}^{p,w_2}$. Moreover, $\|[b,S]f\|_{\mathcal{M}_{\psi_2}^{p,w_2}} \le C\|b\|_*\|f\|_{\mathcal{M}_{\psi_1}^{p,w_1}}$ for $f \in \mathcal{M}_{\psi_1}^{p,w^p}$.

For commutators of singular operators, the strong-type estimate generally fails at the endpoint $p = 1$. To address this, we provide a weak-type endpoint estimate utilizing the Orlicz space structure ($L \log L$), as formulated below.

Theorem 1.8 *Let $(w_1, w_2) \in A_1$, $w_2 \in A_1$ and $w_1 \in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{w_1(B(a,t))}{w_2(B(a,t))} \inf_{t\le s<\infty} \psi_1(a,s) \frac{dt}{t} < \infty. \tag{1.10}$$

If $b \in BMO$, $\Phi(t) = t(1 + \log^+ t)$, and S satisfies

$$w_2(\{x \in \mathbb{R}^n : |[b, T_0](f)(x)| > \sigma\}) \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\sigma}\right) w_1(x) dx \tag{1.11}$$

where $C > 0$ is independent of f and σ, then there exists a constant $C > 0$ such that

$$\|[b, T_0](f)\|_{W\mathcal{M}^{1,w_2}_{\psi_2}} \leq C\|b\|_* \sup_{\sigma>0} \sigma \left\| \Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{L\log L, w_1}_{\psi_1}}.$$

Remark 1.9 If $w_1 = w_2 \in A_1$, $b \in BMO$, then the generalized Calderón-Zygmund operator T_θ which is introduced by Yabuta [58] satisfies Eq. 1.11.

The final part of our results adapts the singular operator cases ($\alpha = 0$) to the mixed-Morrey spaces. We first state the theorem for the base operator T_0.

Theorem 1.10 *Let ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$, $1 \leq p_0 < \infty$, $1 \leq p < \infty$, $(w_1, w_2) \in A_{p/s}$ and $w_2 \in A_{p/s}$ where $1 \leq s < p$ for $1 < p < \infty$ and $s = 1$ for $p = 1$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.8. If T_0 is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$ and from L^{1,w_1} to WL^{1,w_2}, then T_0 is bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_2}_{\psi_2})$ for $1 < p < \infty$ and from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{1,w_1}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, W\mathcal{M}^{1,w_2}_{\psi_2})$.*

Subsequently, we expand this to encompass the commutator $[b, T_0]$ acting on the mixed spaces.

Theorem 1.11 *Let ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$, $1 \leq p_0 < \infty$, $1 \leq p < \infty$, $(w_1, w_2) \in A_{p/s}$ and $w_2 \in A_{p/s}$ where $1 < s < p$ for $1 < p < \infty$ and $s = 1$ for $p = 1$, and $w_1 \in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.9. If $b \in BMO$ and $[b, T_0]$ is bounded from L^{p,w_1} to L^{p,w_2}, then $[b, T_0]$ is bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{q,w_2}_{\psi_2})$. Moreover, for $f \in \mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1}_{\psi_1})$, $\|[b, T_0]f\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,\mathcal{M}^{p,w_2}_{\psi_2})} \leq C\|b\|_*\|f\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,\mathcal{M}^{p,w_1}_{\psi_1})}$.*

Finally, to close our main results, we present the endpoint weak-type estimate for the commutator in the mixed space setting, adopting the same Orlicz space approach used in Theorem 1.8.

Theorem 1.12 *Let $T > 0$, ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$, $1 \leq p_0 < \infty$, $(w_1, w_2) \in A_1$, $w_2 \in A_1$, and $w_1 \in A_\infty$. Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.10. If $b \in BMO$ and T_0 satisfies*

$$w_2(\{x \in \mathbb{R}^n : |[b, T_0](f)(x,t)| > \sigma\}) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x,t)|}{\sigma}\right) w_1(x)dx, \quad t \in (0, T)$$

where $C > 0$ is independent of f and σ then there exists a constant $C > 0$ such that for any suitable function f, we have

$$\|[b, T_0](f)\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,W\mathcal{M}^{1,w_2}_{\psi_2})} \leq C\|b\|_* \sup_{\sigma>0} \sigma \left\| \Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,\mathcal{M}^{L\log L, w_1}_{\psi_1})}.$$

Remark 1.13 Our results generalize the results in [22, 24–27, 29, 30, 33, 46, 47, 56, 57] among others.

Preliminaries: Definitions and Lemmas

In this section, we discuss A_p and $A_{p,q}$ weights and the weighted Lebesgue spaces. We also present the definitions of generalized weighted Morrey spaces, generalized weighted weak Morrey spaces, generalized weighted Morrey spaces of $L\log L$ type, generalized weighted mixed-Morrey spaces, generalized weighted mixed-Morrey spaces, generalized weighted weak mixed-Morrey spaces, and generalized weighted mixed-Morrey spaces of $L\log L$ type. At the end of this section, some lemmas that we shall use to prove the main results about the boundedness of sublinear operators on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces are provided.

A weight w is a nonnegative locally integrable function on $\mathbb{R}^n$ taking values in the interval $(0, \infty)$ almost everywhere. For a weight w and any measurable set $E \subseteq \mathbb{R}^n$, we write $w(E) = \int_E w(x)dx$. The weights that we discuss in this paper belong to the Muckenhoupt class A_p or $A_{p,q}$.

Definition 2.1 [18] For $1 < p < \infty$, we denote by A_p the set of all weights w on $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that for $(a, r) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx\right)\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C.$$

For $p = 1$, we denote by A_1 the set of all weights w for which there exists a constant $C > 0$ such that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx \le C\|w\|_{L^\infty(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

We define $A_\infty := \cup_{p\ge 1} A_p$.

Remark 2.2 Note that the last inequality is equivalent to

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx)\right) \cdot \|w^{-1}\|_{L^\infty(B(a,r))} \le C, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Theorem 2.3 *[18] For each $1 \le p < \infty$ and $w \in A_p$, there exists $C > 0$ such that $w(B)/w(E) \le C(|B|/|E|)^p$ for every ball B and measurable sets $E \subseteq B$.*

Definition 2.4 For $1 < p < \infty$, the pair of weights (w_1, w_2) belongs to A_p if there exists a constant $C > 0$ such that for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w_2(x)dx\right)\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w_1(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C.$$

For $p = 1$, the pair of weights (w_1, w_2) belongs to A_1 if there exists a positive constant $C > 0$ such that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} w_2(x)dx \le C\|w_1\|_{L^\infty(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Associated to a weight $w \in A_p$ with $1 \le p < \infty$, the weighted Lebesgue space $L^{p,w}(\Omega)$ on a measurable subset Ω is the set of all measurable functions f on Ω for which

$$\|f\|_{L^{p,w}(\Omega)} := \left(\int_\Omega |f(x)|^p w(x)dx\right)^{\frac{1}{p}} < \infty.$$

In addition, $WL^{p,w}(\Omega)$ consists of all measurable functions f on Ω for which

$$\|f\|_{WL^{p,w}} := \sup_{\gamma>0} \gamma w(\{x \in \Omega : |f(x)| > \gamma\})^{\frac{1}{p}} < \infty.$$

If $\Omega = \mathbb{R}^n$, we write $L^{p,w} = L^{p,w}(\mathbb{R}^n)$ and $WL^{p,w} = WL^{p,w}(\mathbb{R}^n)$. Notice that if w is constant a.e., then we find that $L^{p,w} = L^p$ and $WL^{p,w} = WL^p$.

Moreover, we denote $L^{p,w}_{\mathrm{loc}}$ by the set of any measurable function f such that $\|f \cdot \chi_{B(a,r)}\|_{L^{p,w}}$ is finite for any ball B and $WL^{p,w}_{\mathrm{loc}}$ by the set of any measurable function f such that $\|f \cdot \chi_{B(a,r)}\|_{WL^{p,w}}$ is finite for any ball B.

Related to the discussion of fractional integral operators I_α, we have another class of weights denoted by $A_{p,q}$.

Definition 2.5 [42, 52] Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. We denote by $A_{p,q}$ the collection of all weights w satisfying

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-p'}dx\right)^{\frac{1}{p'}} \le C$$

for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$ where C is a constant independent of a and r. For $p = 1$ and $q > 1$, we denote by $A_{1,q}$ the collection of all weights w for which there exists a constant $C > 0$ such that for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{1/q} \le C\|w\|_{L^\infty(B(a,r))}$$

One may observe that $w \in A_{p,q}$ if and only if $w^q \in A_{q/p'+1}$ for $1 \le p < q < \infty$ (see [9]). Moreover, we have the following lemma.

Lemma 2.6 *[39, Theorem 3.2.2] Let $1 \le p < q < \infty$. If $w \in A_{p,q}$, then $w^p \in A_p$ and $w^q \in A_q$.*

Definition 2.7 [3] Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. The pair of weights (w_1, w_2) belongs to $A_{p,q}$ if for some $C > 0$ it holds for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$ that

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w_2(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w_1(x)^{-p'}dx\right)^{\frac{1}{p'}} \le C.$$

For $p = 1$, the pair of weights (w_1, w_2) belongs to $A_{1,q}$ if there is a constant $C > 0$ such that

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w_2(x)^q dx\right)^{1/q} \le C\|w_1\|_{L^\infty(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

We rewrite the following results of Muckenhoupt [41] for the Hardy-Littlewood maximal operator M, Muckenhoupt and Wheeden [42] for the Riesz potentials or the fractional integral operators I_α, Coifman and Fefferman [6], Garcia-Cuerva and Rubio de Francia [18], and E. Sawyer [53] for the Calderòn-Zgymund operators S on weighted Lebesgue spaces:

Theorem 2.8 *[18] For $1 < p < \infty$, the Hardy-Littlewood maximal operator M is bounded on $L^{p,w}$ if (and only if) $w \in A_p$. Moreover, M is bounded from $L^{1,w}$ to $WL^{1,w}$ if $w \in A_1$.*

Theorem 2.9 *[42] Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Then, the fractional integral operator I_α is bounded from L^{p,w^p} to L^{q,w^q}. Moreover, if $w \in A_{1,p}$ with $1/q = 1 - \alpha/n$, then I_α is bounded from $L^{1,w}$ to WL^{q,w^q}.*

Theorem 2.10 *[6, 18, 53] For $1 < p < \infty$, the Calderón-Zygmund operator S is bounded on $L^{p,w}$ whenever $w \in A_p$. Moreover, S is bounded from $L^{1,w}$ to $WL^{1,w}$ whenever $w \in A_1$.*

The three theorems will imply the boundedness of the three classical operators on generalized weighted Morrey spaces. The boundedness properties of these operators shall be one of the implications of our results.

Let us now recall some basic definitions and facts for Orlicz functions needed for our results, particularly the boundedness for sublinear operators generated by Calderón-Zygmund operator when $p = 1$. Related to Orlicz functions, we recall the definition of Young functions.

Definition 2.11 A function $\Phi : [0,\infty] \to [0,\infty]$ is called a Young function if it is continuous, convex, and strictly increasing with $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. For a Young function Φ and $w \in A_\infty$, we define the Orlicz space $L^{\Phi,w}$ to be the set of all measurable functions f such that $\int_{\mathbb{R}^n}\Phi(k|f(y)|)w(y)dy$ is finite for some $k > 0$. The space $L^{\Phi,w}$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi,w}} := \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\Phi\left(\frac{|f(y)|}{\lambda}\right)w(y)dy \le 1\right\}.$$

For a measurable subset Ω in $\mathbb{R}^n$, the space $L^{\Phi,w}(\Omega)$ is defined to be the set of all functions f such that

$$\|f\|_{L^{\Phi,w}(\Omega)} = \inf\left\{\lambda > 0 : \frac{1}{w(\Omega)}\int_\Omega \Phi\left(\frac{|f(y)|}{\lambda}\right)w(y)dy \le 1\right\} < \infty.$$

Moreover, we denote by $L^{\Phi,w}_{\mathrm{loc}}$ the set of all measurable functions f such that $\|f\cdot\chi_{B(a,r)}\|_{L^{\Phi,w}} < \infty$ for any ball B.

Given a Young function Φ, we denote by $\tilde{\Phi}$ the complementary Young function for Φ. The following inequality holds for any ball B in $\mathbb{R}^n$ (see [48]).

$$\frac{1}{w(B)}\int_B |f(x)g(x)|w(x)dx \le 2\|f\|_{L^{\Phi,w}(B)}\|f\|_{L^{\tilde{\Phi},w}(B)}. \tag{2.1}$$

An important example of Young function is $\Phi(t) = t(1+\log^+ t)$ where $\log^+ t = \max(0, \log t)$. Its complementary Young function is $\tilde{\Phi} = e^t - 1$. This function will be used in this paper. If $\Phi(t) = t(1+\log^+ t)$, we write

$$\|f\|_{L^{\Phi,w}(B)} = \|f\|_{L\log L^w(B)}, \quad \|f\|_{L^{\tilde{\Phi},w}(B)} = \|f\|_{\exp L^w(B)}.$$

Moreover, $f \in L^{\Phi,w}_{\mathrm{loc}}$ means $f \in L^{\Phi,w}_{\mathrm{loc}}(B) < \infty$ for any ball B. Note that by the fact that $t \le \Phi(t)$ for $t > 0$ and the above definition, we have

$$\|f\|_{L^{1,w}(B)} = \int_B |f(x)|w(x)dx = w(B)\cdot\frac{1}{w(B)}\int_B |f(x)|w(x)dx \le \|f\|_{L\log L^w(B)} \tag{2.2}$$

holds for any ball B.

We now present the definition of the generalized weighted Morrey spaces and the generalized weighted weak Morrey spaces which will become the spaces of our interest in this article. We also define the generalized weighted Morrey spaces Log type. The spaces will be used for boundedness of S_α and $S = S_0$ for the case $p = 1$. Moreover, we also introduce the generalized weighted mixed-Morrey spaces.

Definition 2.12 Let $1 \le p < \infty$, $w \in A_p$, and ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$. The *generalized weighted Morrey space* $\mathcal{M}_\psi^{p,w}$ is the set of all functions $f \in L_{\text{loc}}^{p,w}$ such that

$$\|f\|_{\mathcal{M}_\psi^{p,w}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))} < \infty.$$

Meanwhile, the *generalized weighted weak Morrey space* $W\mathcal{M}_\psi^{p,w}$ is the set of all functions $f \in WL_{\text{loc}}^{p,w}$ such that

$$\|f\|_{W\mathcal{M}_\psi^{p,w}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{WL^{p,w}(B(a,r))} < \infty.$$

Remark 2.13 There are some other variations of generalized weighted Morrey spaces and generalized weighted weak Morrey spaces and their specific conditions, as well as their relations with some classical operators such as in [14–16, 43–45].

Definition 2.14 Let $w \in A_1$ and ψ be a positive function defined on $\mathbb{R}^n \times \mathbb{R}^+$. The *generalized weighted weak Morrey space of log-type* $\mathcal{M}_\psi^{L\log L,w}$ is the set of all functions $f \in L_{\text{loc}}^{\Phi,w}$ such that

$$\|f\|_{\mathcal{M}_\psi^{L\log L,w}} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \|f\|_{L\log L^w(B(a,r))} < \infty.$$

Definition 2.15 Let $1 \le p, q < \infty$, $w \in A_p$, and ψ be a positive function defined on $\mathbb{R}^n \times \mathbb{R}^+$. Suppose that $T > 0$. The *generalized weighted mixed-Morrey space* $\mathcal{M}_\psi^{q,w}(0, T, \mathcal{M}_\varphi^{p,w})$ is the set of all functions f on $\mathbb{R}^n \times (0, T)$ such that

$$\|f\|_{\mathcal{M}_\psi^{q,w}(0,T,\mathcal{M}_\varphi^{p,w})} = \sup_{\substack{a\in\mathbb{R}^n, r>0\\ t\in(0,T)}} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{q}}} \left(\int_{(0,T)\cap(t_0-r,t_0+r)} \|f(\cdot,t)\|_{\mathcal{M}_\varphi^{p,w}}^q dt \right)^{\frac{1}{q}} < \infty.$$

Meanwhile, the *generalized weighted mixed-weak Morrey space* $\mathcal{M}_\psi^{q,w}(0, T, W\mathcal{M}_\varphi^{p,w})$ is the set of all functions f on $\mathbb{R}^n \times (0, T)$ such that

$$\|f\|_{\mathcal{M}_\psi^{q,w}(0,T,W\mathcal{M}_\varphi^{p,w})} = \sup_{\substack{a\in\mathbb{R}^n, r>0\\ t_0\in(0,T)}} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{q}}} \left(\int_{(0,T)\cap(t_0-r,t_0+r)} \|f(\cdot,t)\|_{W\mathcal{M}_\varphi^{p,w}}^q dt \right)^{\frac{1}{q}} < \infty.$$

Definition 2.16 Let $1 \le p, q < \infty$, $w \in A_p$, and the positive functions ψ, φ which are defined on $\mathbb{R}^n \times \mathbb{R}^+$. Suppose that $T > 0$. The *generalized weighted mixed-weak Morrey space Log type* $\mathcal{M}_\varphi^{q,w}(0, T, \mathcal{M}_\psi^{L\log L,w})$ is the set of all functions f on $\mathbb{R}^n \times (0, T)$ such that

$$\|f\|_{\mathcal{M}_\varphi^{q,w}(0,T,\mathcal{M}_\psi^{L\log L,w})} = \sup_{\substack{a\in\mathbb{R}^n, r>0\\ t_0\in(0,T)}} \frac{1}{\varphi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{q}}} \left(\int_{(0,T)\cap(t_0-r,t_0+r)} \|f(\cdot,t)\|_{\mathcal{M}_\psi^{L\log L,w}}^q dt \right)^{\frac{1}{q}} < \infty.$$

Remark 2.17 The Definition 2.15 is generalization of Mixed-Morrey spaces as in [46]. It is easy to see that if N is any operator such that

$$\|Nf(\cdot,t)\|_{\mathcal{M}_{\psi_2}^{p,w_2}} \le C\|f(\cdot,t)\|_{\mathcal{M}_{\psi_1}^{q,w_1}}; \quad f \in \mathcal{M}_{\psi_1}^{q,w_1}, t \in (0,T),$$

then N is also bounded from $\mathcal{M}_\psi^{q,w}(0, T, \mathcal{M}_{\psi_1}^{q,w_1}(\Omega))$ to $\mathcal{M}_\psi^{p,w_2}(0, T, \mathcal{M}_{\psi_2}^{p,w_2}(\Omega))$.

For a measurable subset Ω of $\mathbb{R}^n$, write $\|f\|_{\mathcal{M}_\psi^{p,w}(\Omega)} = \|f \cdot \chi_\Omega\|_{\mathcal{M}_\psi^{p,w}}$ and we write similarly for the other type of Morrey spaces. With the definitions, we shall investigate the boundedness of sublinear operators generated by Ries potentials as well as those generated by Calderón-Zygmund operators on these spaces in the next section.

Next, we discuss the space of Bounded Mean Oscillation (BMO) functions, which serves as the symbol space for the commutators investigated in this study. Recall that the space $BMO = BMO(\mathbb{R}^n)$ consists of all locally integrable functions b on $\mathbb{R}^n$ such that

$$\|b\|_* := \sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{|B(a,r)|} \int_{B(a,r)} |b(y) - b_{B(a,r)}| dy < \infty,$$

where $b_{B(a,r)}$ denotes the average value of b over the ball $B(a, r)$, defined as $b_{B(a,r)} = \frac{1}{|B(a,r)|} \int_{B(a,r)} b(y)dy$.

To establish the boundedness of commutators, it is crucial to understand how the averages of a BMO function behave across different scales. The following lemma provides a logarithmic estimate for the growth of these averages, which is instrumental when dealing with the kernel of the operators.

Lemma 2.18 *[36] Let $b \in BMO$. Then, there exists a positive constant C such that $|b_{B(a,r)} - b_{B(a,t)}| \leq C\|b\|_* \ln \frac{t}{r}$ holds for any $0 < 2r < t$ and $a \in \mathbb{R}^n$.*

In many applications, particularly when involving weights from the Muckenhoupt classes, we require a stronger version of the John-Nirenberg inequality. Lemma 2.19 generalizes the BMO property to $L^{p,w}$ estimates, allowing for the comparison of b with its average over balls of different radii.

Lemma 2.19 *[35] Let $0 < p < \infty$, $w \in A_\infty$, and $b \in BMO$. Then, there exists a positive constant $C > 0$ such that for $a \in \mathbb{R}^n$ and $t, r > 0$,*

$$\left(\int_{B(a,t)} |b(y) - b_{B(a,r)}|^p w(y) dy\right)^{\frac{1}{p}} \leq C\|b\|_* w(B(a,t))^{\frac{1}{p}} \left(1 + \left|\ln \frac{t}{r}\right|\right).$$

As a direct consequence of Lemma 2.19, when the radii t and r coincide, we obtain the following localized weighted estimate. This corollary will also be a key tool for proving the boundedness of commutators in generalized weighted Morrey spaces.

Corollary 2.20 *[55] Let $0 < p < \infty$, $w \in A_\infty$, and $b \in BMO$. Then, there exists a positive constant $C > 0$ such that for any ball B on $\mathbb{R}^n$,*

$$\left(\int_B |b(y) - b_B|^p w(y) dy\right)^{\frac{1}{p}} \leq C\|b\|_* w(B)^{\frac{1}{p}}.$$

Furthermore, for results involving sharper endpoint estimates, we utilize the exponential integrability of BMO functions. The following lemma characterizes this relationship under the A_1 weight condition.

Lemma 2.21 *[59] Let $w \in A_1$ and $b \in BMO$. Then, there is a constant $C > 0$ such that $\|b - b_B\|_{\exp L^w(B)} \leq C\|b\|_*$ holds for any ball B on $\mathbb{R}^n$.*

We say that the commutator $[b, S_\alpha]$, generated by an operator S_α and a BMO function b, is bounded from L^{p,w_1} to L^{q,w_2} whenever there exists a constant $C > 0$ such that $\|[b, S_\alpha]f\|_{L^{q,w_2}} \leq C\|b\|_*\|f\|_{L^{p,w_1}}$ for $f \in L^{p,w_1}$. We shall use this notion regarding the commutator.

Suppose that S_α is bounded from L^{p,w_1} to L^{q,w_2} or from L^{p,w_1} to WL^{q,w_2} which satisfies Eq. 1.1 where w_1 and w_2 are certain weights on $\mathbb{R}^n$. Since the operator S_α is given only through the off–support estimate Eq. 1.1, it is not directly defined on functions in the generalized weighted Morrey space and its other variations. For each ball $B(a, r)$, we decompose $f \in L^{p,w_1}_{\mathrm{loc}}$ in the usual way (see [17, 34, 49, 50, 57] for the case of sublinear operators, and [8, 21, 23, 28, 31, 32] for the case of linear operators):

$$f = f\chi_{B(a,2r)} + f\chi_{B(a,2r)^c}.$$

Under the assumption that $S_\alpha(f)(x)$ is well defined, by the sublinearity of S_α, we then have

$$|S_\alpha(f)(x)| \leq |S_\alpha\big(f\chi_{B(a,2r)}\big)(x)| + |S_\alpha\big(f\chi_{B(a,2r)^c}\big)(x)|, \qquad x \in B(a, r) \tag{2.3}$$

The local component belongs to L^{p,w_1}, while the global part is controlled precisely by the estimate Eq. 1.1. With this observation, the commutator $[b, S_\alpha]$ is well defined from L^{p,w_1} to L^{q,w_2} under some assumptions. As in [57], it is also defined on the generalized weighted Morrey space, as stated in Theorem 1.1.

We next consider the operator T_α and its commutator on generalized weighted mixed-Morrey spaces. Let $T > 0$. Following the same approach as in the context of generalized weighted Morrey spaces, we write

$$|T_\alpha(f)(x,t)| \leq |T_\alpha\big(f\chi_{B(a,2r)}\big)(x,t)| + |T_\alpha\big(f\chi_{B(a,2r)^c}\big)(x,t)|, \qquad x \in B(a,r), t \in (0, T),$$

see [46] for example. We say T_α and $[b, T_\alpha]$ are bounded from L^{p,w_1} to L^{q,w_2} if there exists $C > 0$ such that

$$\|T_\alpha(f)(\cdot,t)\|_{L^{q,w_2}} \leq C\|f(\cdot,t)\|_{L^{p,w_1}}, \quad f \in L^{p,w_1}, t \in (0, T)$$

and

$$\|[b, T_\alpha](f)(\cdot,t)\|_{L^{q,w_2}} \leq C\|b\|_*\|f(\cdot,t)\|_{L^{p,w_1}}, \quad f \in L^{p,w_1}, t \in (0, T).$$

Related to the weak-type, we say that T_α is bounded from L^{p,w_1} to WL^{q,w_2} if there exists a constant $C > 0$ such that

$$\|T_\alpha(f)(\cdot,t)\|_{WL^{q,w_2}} \leq C\,\|f(\cdot,t)\|_{L^{p,w_1}}$$

for all $f \in L^{p,w_1}$ and $t \in (0, T)$. Such boundedness for certain operators is discussed in [46], building on results from [1, 5, 6].

We then end this section by providing some lemmas which will be used later.

Lemma 2.22 *Suppose that u_1, u_2, v_1, and v_2 are positive functions on $\mathbb{R}^n \times \mathbb{R}^+$ and $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Suppose that $g : \mathbb{R}^n \times \mathbb{R}^+ \to [0, \infty)$ is increasing. If*

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{v_2(a,r)} \int_r^\infty h(t,r) \frac{\inf_{t\le s<\infty} u_1(a,l)v_1(a,l)}{u_2(a,t)} \frac{dt}{t} \le C < \infty$$

and

$$\sup_{(a,t)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{g(a,t)}{u_1(a,t)v_1(a,t)} < \infty \tag{2.4}$$

then

$$\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \int_r^\infty h(t,r)u_2(a,t)^{-1}g(a,t)\frac{dt}{t} \le Cv_2(a,r) \sup_{(a,t)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{g(a,t)}{u_1(a,t)v_1(a,t)}.$$

Proof We may see that the quantity $\inf_{t\le l<\infty} u_1(a,l)v_1(a,l)$ is always positive by the condition (2.4). Therefore, the assumptions allow us to have that

$$\begin{aligned}
&\int_r^\infty h(t,r)u_2(a,t)^{-1}g(a,t)\frac{dt}{t} \\
&= \int_r^\infty h(t,r)\frac{\inf_{t\le l<\infty} u_1(a,l)v_1(a,l)}{\inf_{t\le l<\infty} u_1(a,l)v_1(a,l)} u_2(a,t)^{-1}g(a,t)\frac{dt}{t} \\
&= \int_r^\infty h(t,r)\frac{\inf_{t\le l<\infty} u_1(a,l)v_1(a,l)}{u_2(a,t)} \sup_{t\le l<\infty} \frac{1}{u_1(a,l)}\frac{1}{v_1(a,l)} g(a,l)\frac{dt}{t} \\
&= \int_r^\infty h(t,r)\frac{\inf_{t\le l<\infty} u_1(a,l)v_1(a,l)}{u_2(a,t)}\frac{dt}{t} \cdot \sup_{(a,t)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{g(a,t)}{u_1(a,t)v_1(a,t)} \\
&\le Cv_2(a,r) \sup_{(a,t)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{g(a,t)}{u_1(a,t)v_1(a,t)}
\end{aligned}$$

for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$. This proves the lemma. □

Lemma 2.23 *Let w be a weight on $\mathbb{R}^n$, $0 < \alpha < n$, $1 \le p < n/\alpha$, and $1/q = 1/p - \alpha/n$. Let $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Then,*

$$r^{-\frac{\alpha}{n}+\frac{1}{s}} \le w_2(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+,$$

for $1 < p < n/\alpha$, and

$$r^{-\frac{\alpha}{n}+1} \le w_2(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^\infty(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Proof First, suppose that $1 < p < \infty$. Since $p < q$ and $p' < s/(p-s)$, by Hölder's inequality, we obtain that

$$\begin{aligned}
1 &= \frac{1}{|B(a,r)|}\int_{B(a,r)} dx = \frac{1}{|B(a,r)|}\int_{B(a,r)} w_2(y)\cdot w_2(y)^{-1}dy \\
&\le \frac{1}{B|(a,r)|}\left(\int_{B(a,r)} w_2^p(y)dy\right)^{\frac{1}{p}} \left(\int_{B(a,r)} w_2^{-p'}(y)dy\right)^{\frac{1}{p'}} \\
&\le \left(\frac{1}{B|(a,r)|}\int_{B(a,r)} w_2^p(y)dy\right)^{\frac{1}{p}} \left(\frac{1}{B|(a,r)|}\int_{B(a,r)} w_2^{-p'}(y)dy\right)^{\frac{1}{p'}} \\
&\le \left(\frac{1}{B|(a,r)|}\int_{B(a,r)} w_2^q(y)dy\right)^{\frac{1}{q}} \left(\frac{1}{B|(a,r)|}\int_{B(a,r)} w_2^{-\frac{ps}{p-s}}(y)dy\right)^{\frac{p-s}{ps}},
\end{aligned}$$

which implies that

$$|B(a,r)|^{\frac{1}{q}+\frac{1}{s}-\frac{1}{p}} \le w_2(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,r))}$$

and

$$r^{-\frac{\alpha}{n}+\frac{1}{s}} \le w_2(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

For $p = 1$, we have

$$1 = \frac{1}{|B(a,r)|} \int_{B(a,r)} \leq \frac{1}{B|(a,r)|} \left(\int_{B(a,r)} w_2(y)dy \right) \|w_2^{-1}\|_{L^\infty(B(a,r))}$$
$$\leq \frac{1}{B|(a,r)|} \left(\int_{B(a,r)} w_2(y)^q dy \right)^{\frac{1}{q}} \|w_2^{-1}\|_{L^\infty(B(a,r))},$$

and so

$$r^{-\frac{\alpha}{n}+1} \leq w_2^q(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^\infty(B(a,r))}, \quad (a,r) \in \mathbb{R}^n \times \mathbb{R}^+,$$

which completes the proof of the lemma. □

Proof of Theorem 1.2–1.5

The estimates given in Eqs. 1.1 and 1.4 play a central role throughout this paper. Consequently, in the proofs of our main theorems in this section and the "Proof of Theorem 1.6–1.8 and Theorem 1.10–1.12” section, we must first verify that the right-hand sides of these estimates are finite for any function f belonging to generalized weighted Morrey spaces in the hypothesis. This initial verification is particularly crucial when estimating the terms $S_\alpha(f_2)$ and $S_\alpha((b - b_B) f_2)$ over a ball $B \subset \mathbb{R}^n$.

The primary focus of this section is to establish the boundedness of sublinear operators generated by Riesz potentials, as well as their commutators with BMO functions, on generalized weighted Morrey spaces with different weights (namely Theorems 1.2, 1.4, 1.3, and 1.5). Before proceeding to the proofs of these main results, we first establish the following preliminary estimates.

Proposition 3.1 *Let $0 < \alpha < n$, $1 \leq p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \leq s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. If S_α is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then there exists a constant $C > 0$ such that for every $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$ and $f \in L^{p,w_1^p}_{\mathrm{loc}}$,*

$$\|S_\alpha f\|_{L^{q,w_2^q}(B(a,r))} \leq C_1 w_2^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w_2^q(B(a,t))^{-\frac{1}{q}} \|f\|_{L^{p,w_1^p}B(a,t)} \frac{dt}{t}, \tag{3.1}$$

for $1 < p < n/\alpha$, and

$$\|S_\alpha f\|_{WL^{q,w_2^q}(B(a,r))} \leq C_2 w_2^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w_2^q(B(a,t))^{-\frac{1}{q}} \|f\|_{L^{1,w_1}B(a,t)} \frac{dt}{t} \tag{3.2}$$

Remark 3.2 We shall see later in the proof of Theorem 1.2 that the right-hand side of Eqs. 3.1 and 3.2 is finite if f belongs to some generalized weighted Morrey spaces under some assumptions.

Proof *(Proof of Proposition 3.1)* Given $a \in \mathbb{R}^n$ and $r > 0$, we decompose f as $f := f_1 + f_2$ where $f_1 := f \cdot \chi_{B(a,2r)}$. Note that $f_1 \in L^{p,w_1^p}$. Now, since S_α is bounded from L^{p,w_1^p} to L^{q,w_2^q} and from $L^{1,w}$ to WL^{q,w^q},

$$\|S_\alpha f_1\|_{L^{q,w_2^q}(B(a,r))} \leq \|S_\alpha f_1\|_{L^{q,w_2^q}} \leq C\|f_1\|_{L^{p,w_1^p}} = C\|f\|_{L^{p,w_1^p}(B(a,2r))}$$

for $1 < p < n/\alpha$, and

$$\|S_\alpha f_1\|_{WL^{q,w_2^q}(B(a,r))} \leq \|S_\alpha f_1\|_{WL^{q,w_2^q}} \leq C\|f_1\|_{L^{1,w_1}} = C\|f\|_{L^{1,w_1}(B(a,2r))}.$$

Thus, by Lemma 2.23 and the assumption that $w_2^s \in A_{p/s,q/s}$ for $1 < p < n/\alpha$, we have

$$\begin{aligned}
\|f\|_{L^{p,w_1^p}(B(a,2r))} &= C r^{\frac{n}{s}-\alpha} \|f\|_{L^{p,w^p}(B(a,2r))} \int_{2r}^\infty \frac{t}{t^{n/s-\alpha+1}} \\
&\leq C w_2^q(B(a,r))^{\frac{1}{q}} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,r))} \int_{2r}^\infty \|f\|_{L^{p,w^p}(B(a,t))} \frac{dt}{t^{n/s-\alpha+1}} \\
&\leq C w_2^q(B(a,r))^{\frac{1}{q}} \int_{2r}^\infty \|f\|_{L^{p,w^p}(B(a,t))} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,t))} \frac{dt}{t^{n/s-\alpha+1}} \\
&\leq C w_2^q(B(a,r))^{\frac{1}{q}} \int_{2r}^\infty \|f\|_{L^{p,w^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}.
\end{aligned}$$

Hence,

$$\|S_\alpha f_1\|_{L^{q,w_2^q}(B(a,r))} \leq C w_2^q(B(a,r))^{\frac{1}{q}} \int_{2r}^\infty \|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}$$

for $1 < p < n/\alpha$, and

$$\|S_\alpha f_1\|_{L^{q,w_2^q}(B(a,r))} \le C w_2^q(B(a,r))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L^{1,w_1}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}.$$

Next, we consider $S_\alpha(f_2)$. Based on the previous remark, at this moment we may assume that the right-hand side of Eqs. 3.1 and 3.2 is finite. We thus need to check that the right-hand side of Eq. 1.1 is also finite using this assumption. If we achieve this, we may use Eq. 1.1 in estimating $S_\alpha(f_2)$.

Suppose first that $1 < s < p < n/\alpha$. Then, Hölder's inequality implies

$$\begin{aligned}\left(\int_{B(a,2^{k+1}r)} |f(y)|^s dy\right)^{\frac{1}{s}} &= \left(\int_{B(a,2^{k+1}r)} |f(y)|^s w_1(y)^s w_1(y)^{-s} dy\right)^{\frac{1}{s}} \\ &= \|f\|_{L^{p,w_1^p}(B(a,2^{k+1}r))} \|w_1^{-1}\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))}.\end{aligned}$$

Hence, by the assumption $(w_1^s, w_2^s) \in A_{p/s,q/s}$, we obtain that

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{1}{|B(a,2^{k+1}r)|^{\frac{1}{s}-\frac{\alpha}{n}}} \left(\int_{B(a,2^{k+1}r)} |f(y)|^s dy\right)^{\frac{1}{s}} \\ &\quad\le C \sum_{k=1}^{\infty} \left(2^{k+1}r\right)^{\alpha-\frac{n}{s}} \|f\|_{L^{p,w_1^p}(B(a,2^{k+1}r))} \|w_1^{-1}\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))} \\ &\quad= C \sum_{k=1}^{\infty} \int_{2^{k+1}r}^{2^{k+2}r} \|f\|_{L^{p,w_1^p}(B(a,t))} \|w_1^{-1}\|_{L^{ps/(p-s)}(B(a,t))} \frac{dt}{t^{-\alpha+\frac{n}{s}+1}} \\ &\quad\le C \int_{2r}^{\infty} \|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned}$$

Thus, the right side of the inequality Eq. 1.1 is finite. Consequently, we have:

$$\begin{aligned}|S_\alpha f_2(x)| &\le \sup_{x\in B(a,r)} \left|S_\alpha\left(f\cdot\chi_{B(a,2r)^c}\right)(x)\right| \\ &\le C \sum_{k=1}^{\infty} (2^{k+1}r)^{-\frac{n}{s}+\alpha} \left(\int_{B(a,2^{k+1}r)} |f(y)|^s dy\right)^{1/s} \\ &\le C \int_{2r}^{\infty} \|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}.\end{aligned}$$

for $x \in B(a,r)$ and we obtain

$$\|S_\alpha f_2\|_{L^{q,w_2^q}(B(a,r))} \le C w_2^q(B(a,r))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}} \frac{dt}{t}$$

for $1 < p < n/\alpha$. Next, for $p = s = 1$

$$\left(\int_{B(a,2^{k+1}r)} |f(y)|^s dy\right)^{\frac{1}{s}} \le \|f\|_{L^{1,w_1}(B(a,2^{k+1}r))} \|w_1^{-1}\|_{L^\infty(B(a,2^{k+1}r))}.$$

Therefore,

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{1}{|B(a,2^{k+1}r)|^{1-\frac{\alpha}{n}}}\int_{B(a,2^{k+1}r)}|f(y)|dy\\
&\quad\le C\sum_{k=1}^{\infty}\frac{1}{|B(a,2^{k+1}r)|^{1-\frac{\alpha}{n}}}\|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^\infty(B(a,2^{k+1}r))}\\
&\quad\le C\sum_{k=1}^{\infty}\left(2^{k+1}r\right)^{\alpha-n}\|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^\infty(B(a,2^{k+1}r))}\\
&\quad= C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\frac{dt}{t^{-\alpha+n+1}}\cdot\|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^\infty(B(a,2^{k+1}r))}\\
&\quad\le C\int_{2r}^{\infty}\|f\|_{L^{1,w_1}(B(a,t))}w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}.
\end{aligned}$$

Hence, the right side of the inequality Eq. 1.1 is finite for $p=1$.

We thus have

$$\begin{aligned}
\|S_\alpha f_2\|_{WL^{q,w_2^q}(B(a,r))}&\le\|S_\alpha f_2\|_{L^{q,w_2^q}(B(a,r))}\\
&\le Cw_2^q(B(a,r))^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L^{1,w_1}(B(a,t))}w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}.
\end{aligned}$$

By Eq. 2.3, we obtain

$$\|S_\alpha f\|_{L^{q,w_2^q}(B(a,r))}\le Cw_2^q(B(a,r))^{\frac{1}{q}}\int_r^{\infty}w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}B(a,t)}\frac{dt}{t}$$

for $1<p<n/\alpha$, and

$$\|S_\alpha f\|_{WL^{q,w_2^q}(B(a,r))}\le Cw_2^q(B(a,r))^{\frac{1}{q}}\int_r^{\infty}w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{1,w_1}B(a,t)}\frac{dt}{t},$$

as desired. □

Proposition 3.3 *Let $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $(w_1^s,w_2^s)\in A_{p/s,q/s}$ and $w_2^s\in A_{p/s,q/s}$ where $1\le s<p$, and $w_1\in A_\infty$. If $[b,S_\alpha]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q}, then there exists a constant $C>0$ such that for every $(a,r)\in\mathbb{R}^n\times\mathbb{R}^+$ and $f\in L^{p,w_1^p}_{\text{loc}}$,*

$$\|[b,S_\alpha]f\|_{L^{q,w_2^q}(B(a,r))}\le C\|b\|_*w_2^q(B(a,r))^{\frac{1}{q}}\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}B(a,t)}\frac{dt}{t}\tag{3.3}$$

Remark 3.4 We shall see in the proof of Theorem 1.3 that the right-hand side of Eq. 3.3 is finite if f belongs to some generalized weighted Morrey spaces under some conditions.

Proof of Proposition 3.3 Let $a\in\mathbb{R}^n$ and $r>0$, and we represent the function $f\in L^{p,w_1^p}_{\text{loc}}$ as $f:=f_1+f_2$ where $f_1:=f\cdot\chi_{B(a,2r)}$. Since $f_1\in L^{p,w_1^p}$ and $[b,S_\alpha]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q}, we find that

$$\|[b,S_\alpha]f_1\|_{L^{q,w_2^q}(B(a,r))}\le\|[b,S_\alpha]f_1\|_{L^{q,w_2^q}}\le C\|b\|_*\|f_1\|_{L^{p,w_1^p}}=C\|b\|_*\|f\|_{L^{p,w_1^p}(B(a,2r))}$$

and by Lemma 2.23 as in Proposition 3.1,

$$\|[b,S_\alpha]f_1\|_{L^{q,w_2^q}(B(a,r))}\le C\|b\|_*w_2^q(B(a,r))^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L^{p,w_1^p}(B(a,t))}w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}\tag{3.4}$$

For f_2, from Eq. 1.2 and by the sublinearity of S_α, we then have that

$$|[b,S_\alpha]f_2(x)|\le C|b(x)-b_{B(a,r)}||S_\alpha(f_2)(x)|+|S_\alpha((b_{B(a,r)}-b)(f))(x)|:=J_1(x)+J_2(x),$$

for $x \in B(a,r)$ where $J_1(x)$ makes sense if the right-hand side of Eq. 1.1 is finite and $J_2(x)$ makes sense if the right-hand side of the inequality:

$$\sup_{x\in B(a,r)} \left|S_\alpha\left((b_{B(a,r)}-b)(f)\cdot \chi_{B(a,2r)^c}\right)(x)\right| \\ \leq C\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|(b(y)-b_{B(a,r)})f(y)|^s dy\right)^{1/s} \tag{3.5}$$

is also finite. Based on the previous remark, we may assume that the right-hand side of Eq. 3.3 is finite. Hence, as in Proposition 3.1, we have that the right-hand side of Eq. 1.1 for f_2 is finite. Moreover,

$$J_1(x) \leq C|b(x)-b_{B(a,r)}|\int_{2r}^{\infty}\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}, \quad x\in B(a,r).$$

Since $w_2^s \in A_{p/s,q/s}$, we have that $w_2^q \in A_{q/s}$ and by Lemma 2.20, it follows that

$$\|J_1\|_{L^{q,w_2^q}(B(a,r))} \leq C\|b\|_* w_2^q(B(a,r))^{\frac{1}{q}}\int_{2r}^{\infty}\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t} \tag{3.6}$$

Next, we shall see that the right-hand side of Eq. 3.5 is finite by showing that it is bounded by the right-hand side of Eq. 3.3. Using Hölder's inequality, we first deduce that

$$\left(\int_{B(a,2^{k+1}r)}|(b(y)-b_{B(a,r)})f(y)|^s dy\right)^{\frac{1}{s}} \\ \leq \|f\|_{L^{p,w_1^p}(B(a,2^{k+1}r))}\left\|(b-b_{B(a,r)})w_1^{-1}\right\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))}.$$

Thus, by the assumption $w_1 \in A_\infty$ and $(w_1^s, w_2^s) \in A_{p/s,q/s}$, the last inequality, and Lemma 2.23, we have

$$\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|(b(y)-b_{B(a,r)})f(y)|^s dy\right)^{\frac{1}{s}} \\ \leq C\int_{2r}^{\infty}\|f\|_{L^{p,w_1^p}(B(a,t))}\|(b-b_{B(a,r)})w_1^{-1}\|_{L^{\frac{ps}{p-s}}(B(a,t))}\frac{dt}{t^{-\alpha+n/s+1}} \\ \leq C\int_{2r}^{\infty}\|f\|_{L^{p,w_1^p}(B(a,t))}\|b\|_*\left(1+\ln\frac{t}{r}\right)\left(w_1^{-\frac{ps}{p-s}}(B(a,t))\right)^{\frac{p-s}{ps}}\frac{dt}{t^{-\alpha+n/s+1}} \\ \leq C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}.$$

Hence, the right-hand side of Eq. 3.5 is finite, as claimed. Further, we get

$$J_2(x) \leq \sup_{x\in B(a,r)}\left|S_\alpha\left((b_{B(a,r)}-b)(f)\cdot\chi_{B(a,2r)^c}\right)(x)\right| \\ \leq C\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}+\alpha}\left(\int_{B(a,2^{k+1}r)}|(b(y)-b_{B(a,r)})f(y)|^s dy\right)^{\frac{1}{s}} \\ \leq C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t}$$

for $x \in B(a,r)$. It follows that

$$\|J_2\|_{L^{q,w_2^q}(B(a,r))} \leq C w_2^q(B(a,r))^{\frac{1}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t} \tag{3.7}$$

By combining Eqs. 3.6 and 3.7, we then obtain

$$\|[b,S_\alpha]f_2\|_{L^{q,w_2^q}(B(a,r))} \leq C\|b\|_* w_2^q(B(a,r))^{\frac{1}{q}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1^p}(B(a,t))} w_2^q(B(a,t))^{-\frac{1}{q}}\frac{dt}{t} \tag{3.8}$$

Now Eqs. 3.4 and 3.8 imply Eq. 3.3 which proves the proposition. □

Now, we are ready to prove Theorem 1.2, Theorem 1.3, Theorem 1.4, and Theorem 1.5.

Proof of Theorem 1.2 We set $h(t,r)=1$ for $(t,r)\in\mathbb{R}^+\times R^+$ and

$$u_1(a,r)=w_1^p(B(a,r))^{\frac{1}{p}},\quad v_1(a,r)=\psi_1(x,r)$$
$$u_2(a,r)=w_2^q(B(a,r))^{\frac{1}{q}},\quad v_2(a,r)=\psi_2(a,r)$$

for $(x,r)\in\mathbb{R}^n\times\mathbb{R}^+$ and use Lemma 2.22 to have the following inequality for $(x,r)\in\mathbb{R}^d\times\mathbb{R}^+$:

$$\frac{1}{\psi_2(a,r)}\int_r^\infty w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}(B(a,t))}\frac{dt}{t}\le C\|f\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}}$$

By Proposition 3.1, assumptions, and the last inequality, we obtain that

$$\begin{aligned}\|S_\alpha f\|_{\mathcal{M}^{q,w^q}_{\psi_2}}&=\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\frac{1}{w_2^q(B(a,r))^{\frac{1}{q}}}\|S_\alpha f\|_{L^{q,w_2^q}(B(a,r))}\\&\le C\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\int_r^\infty w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}(B(a,r))}\frac{dt}{t}\\&\le C\|f\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}}\end{aligned}$$

for $1<p<n/\alpha$. Meanwhile,

$$\begin{aligned}\|S_\alpha f\|_{W\mathcal{M}^{q,w_2^q}_{\psi_2}}&=\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\frac{1}{w_2^q(B(a,r))^{\frac{1}{q}}}\|S_\alpha f\|_{WL^{q,w_2^q}(B(a,r))}\\&\le C\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\int_r^\infty w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,t))}\frac{dt}{t}\\&\le C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.\end{aligned}$$

This proves the theorem. □

Proof of Theorem 1.3 We use a similar argument in the proof of Theorem 1.2. We use 2.22 by first setting $h(t,r)=1+\ln(t/r)$ for $(t,r)\in\mathbb{R}^+\times R^+$ and

$$u_1(a,r)=w_1^p(B(a,r))^{\frac{1}{p}},\quad v_1(x,r)=\psi_1(a,r)$$
$$u_2(a,r)=w_2^q(B(a,r))^{\frac{1}{q}},\quad v_2(x,r)=\psi_2(a,r)$$

for $(x,r)\in\mathbb{R}^n\times\mathbb{R}^+$. The choices of functions allow us to have the following inequality for $(x,r)\in\mathbb{R}^d\times\mathbb{R}^+$:

$$\frac{1}{\psi_2(a,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}(B(a,t))}\frac{dt}{t}\le C\|f\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}}.$$

By Proposition 3.3, assumptions, and the last inequality, we obtain that

$$\begin{aligned}\|[b,S_\alpha]f\|_{\mathcal{M}^{q,w_2^q}_{\psi_2}}&=\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\frac{1}{w_2^q(B(a,r))^{\frac{1}{q}}}\|[b,S_\alpha]f\|_{L^{q,w_2^q}(B(a,r))}\\&\le C\sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+}\frac{1}{\psi_2(a,r)}\int_r^\infty\left(1+\ln\frac{t}{r}\right)w_2^q(B(a,t))^{-\frac{1}{q}}\|f\|_{L^{p,w_1^p}(B(a,r))}\frac{dt}{t}\\&\le C\|f\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}}.\end{aligned}$$

This proves the theorem. □

Proof of Theorem 1.4 One may replace $f(\cdot)$ by $f(\cdot,t)$ as well as S_α by T_α in Proposition 3.1 to obtain

$$\|T_\alpha f(\cdot,t)\|_{\mathcal{M}^{q,w_2^q}_{\psi_2}}\le C\|f(\cdot,t)\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}},\quad f\in\mathcal{M}^{p,w_1^p}_{\psi_1}$$

where $C>0$ is independent of t. Therefore, by Definition 2.15 and Remark 2.17, we have the desired results of T_α on generalized weighted mixed-morrey spaces. □

Proof of Theorem 1.5 One may replace $f(\cdot)$ by $f(\cdot, t)$ as well as S_α by T_α in Proposition 3.3, and obtain

$$\|[b, T_\alpha]f(\cdot, t)\|_{\mathcal{M}^{q,w_2^q}_{\psi_2}} \leq C\|b\|_*\|f(\cdot, t)\|_{\mathcal{M}^{p,w_1^p}_{\psi_1}}, \quad f \in \mathcal{M}^{p,w_1^p}_{\psi_1}$$

where $C > 0$ is independent of t. The conclusion then follows from Definition 2.15 and Remark 2.17. □

Proof of Theorem 1.6–1.8 and Theorem 1.10–1.12

We prove the boundedness for sublinear operators S generated by Calderòn-Zymgund operators on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. In addition, we prove the boundedness for the commutator of the operator with BMO functions on generalized weighted Morrey spaces.

Proposition 4.1 *Let $1 \leq p < \infty$ and (w_1, w_2), $w_2 \in A_{p/s}$, where $1 \leq s < p$ for $1 < p < \infty$ and $s = 1$ for $p = 1$. If S is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$ and from L^{1,w_1} to WL^{1,w_2}, then there exists a constant $C > 0$ such that for every $(a, r) \in \mathbb{R}^n \times \mathbb{R}^+$ and $f \in L^{p,w_1}_{\mathrm{loc}}$,*

$$\|Sf\|_{L^{p,w_2}(B(a,r))} \leq C_1 w_2(B(a, r))^{\frac{1}{p}} \int_r^\infty w_2(B(a, t))^{-\frac{1}{p}} \|f\|_{L^{p,w_1}B(a,t)} \frac{dt}{t}, \tag{4.1}$$

for $1 < p < \infty$, and

$$\|Sf\|_{WL^{1,w_2}(B(a,r))} \leq C_2 w_2(B(a, r)) \int_r^\infty w_2(B(a, t))^{-1} \|f\|_{L^{1,w_1}B(a,t)} \frac{dt}{t}. \tag{4.2}$$

Remark 4.2 We shall see in the proof of Theorem 1.6 that the right-hand side of Eqs. 4.1 and 4.2 is finite if f belongs to some generalized weighted Morrey spaces under some assumptions.

Proof Let $a \in \mathbb{R}^n$ and $r > 0$, and we write $f := f_1 + f_2$ where $f_1 := f \cdot \chi_{B(a,2r)}$. Since S is bounded from L^{p,w_1} to L^{p,w_2} and from L^{1,w_1} to WL^{1,w_2},

$$\|Sf_1\|_{L^{p,w_2}(B(a,r))} \leq \|Sf_1\|_{L^{p,w_2}} \leq C\|f_1\|_{L^{p,w_1}} = C\|f\|_{L^{p,w_1}(B(a,2r))}$$

for $1 < p < \infty$ and

$$\|Sf_1\|_{WL^{1,w_2}(B(a,r))} \leq \|Sf_1\|_{L^{1,w_2}} \leq C\|f_1\|_{L^{1,w_1}} = C\|f\|_{L^{1,w_1}(B(a,2r))}.$$

Thus, by Lemma 2.23 and the assumption that $w_2 \in A_{p/s}$, we have

$$\begin{aligned}
\|f\|_{L^{p,w_1}(B(a,2r))} &= Cr^{\frac{n}{s}} \|f\|_{L^{p,w_1}(B(a,2r))} \int_{2r}^\infty \frac{dt}{t^{n/s+1}} \\
&\leq Cw_2(B(a, r))^{\frac{1}{p}} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,r))} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} \frac{dt}{t^{n/s+1}} \\
&\leq Cw_2(B(a, r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} \|w_2^{-1}\|_{L^{ps/(p-s)}(B(a,t))} \frac{dt}{t^{n/s+1}} \\
&\leq Cw_2(B(a, r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} w_2(B(a, t))^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned}$$

Hence,

$$\|Sf_1\|_{L^{p,w_2}(B(a,r))} \leq Cw_2(B(a, r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} w_2(B(a, t))^{-\frac{1}{p}} \frac{dt}{t} \tag{4.3}$$

for $1 < p < \infty$, and

$$\|Sf_1\|_{WL^{1,w_2}(B(a,r))} \leq w_2(B(a, r)) \int_{2r}^\infty \|f\|_{L^{1,w_1}(B(a,t))} w_2(B(a, t))^{-1} \frac{dt}{t}. \tag{4.4}$$

In view of the previous remark, we may assume the finiteness of the right-hand sides of Eqs. 4.1 and 4.2. Thus, we obtain the estimate Eq. 1.4, provided its right-hand side is finite. First, consider the case where $1 < p < \infty$ and $1 \leq s < p$. Then, Hölder's inequality implies that

$$\begin{aligned}\left(\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{\frac{1}{s}} &= \left(\int_{B(a,2^{k+1}r)}|f(y)|^s w_1(y)^{\frac{s}{p}} w_1(y)^{-\frac{s}{p}} dy\right)^{\frac{1}{s}}\\ &\le \left(\int_{B(a,2^{k+1}r)}\left(|f(y)|^s\right)^{\frac{p}{s}}\left(w_1(y)^{\frac{s}{p}}\right)^{\frac{p}{s}} dy\right)^{\left(\frac{s}{p}\right)\left(\frac{1}{s}\right)}\\ &\quad\times\left(\int_{B(a,2^{k+1}r)}\left(w_1(y)^{-\frac{s}{p}}\right)^{\frac{p}{p-s}} dy\right)^{\left(\frac{p-s}{p}\right)\left(\frac{1}{s}\right)}\\ &= \|f\|_{L^{p,w_1}(B(a,2^{k+1}r))}\|w_1^{-\frac{1}{p}}\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))}.\end{aligned}$$

Hence, by the assumption $(w_1, w_2) \in A_{p/s}$, we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{1}{|B(a,2^{k+1}r)|}\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{\frac{1}{s}}\\ &\le C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\frac{dt}{t^{+\frac{n}{s}+1}}\cdot\|f\|_{L^{p,w_1}(B(a,2^{k+1}r))}\|w_1^{-\frac{1}{p}}\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))}\\ &= C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\|f\|_{L^{p,w_1}(B(a,t))}\|w_1^{-\frac{1}{p}}\|_{L^{ps/(p-s)}(B(a,t))}\frac{dt}{t^{\frac{n}{s}+1}}\\ &\le C\int_{2r}^{\infty}\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}$$

Thus, the right-hand side of inequality Eq. 1.4 is finite and 1.4 valid for $1 \le s < p < \infty$. Moreover,

$$\|Sf_2\|_{L^{p,w_2}(B(a,r))} \le Cw_2(B(a,r))^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t}. \tag{4.5}$$

Next, for the case where $p = 1$ and $s = 1$, we have

$$\left(\int_{B(a,2^{k+1}r)}|f(y)|^s dy\right)^{\frac{1}{s}} \le \|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^{\infty}(B(a,2^{k+1}r))}.$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{|B(a,2^{k+1}r)|}\int_{B(a,2^{k+1}r)}|f(y)|dy &\le C\sum_{k=1}^{\infty}\frac{1}{|B(a,2^{k+1}r)|}\|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^{\infty}(B(a,2^{k+1}r))}\\ &= C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\frac{dt}{t^{n+1}}\cdot\|f\|_{L^{1,w_1}(B(a,2^{k+1}r))}\|w_1^{-1}\|_{L^{\infty}(B(a,2^{k+1}r))}\\ &= C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\|f\|_{L^{1,w_1}(B(a,t))}\|w_1^{-1}\|_{L^{\infty}(B(a,t))}\frac{dt}{t^{n+1}}\\ &\le C\int_{2r}^{\infty}\|f\|_{L^{1,w_1}(B(a,t))}w_2(B(a,t))^{-1}\frac{dt}{t}.\end{aligned}$$

Hence, the estimate Eq. 1.4 is also valid for $p = s = 1$ since its right-hand side is finite. In addition,

$$\begin{aligned}\|Sf_2\|_{WL^{1,w_2}(B(a,r))} &\le \|S_\alpha f_2\|_{L^{1,w_2}(B(a,r))}\\ &\le Cw_2(B(a,r))\int_{2r}^{\infty}\|f\|_{L^{1,w_1}(B(a,t))}w_2(B(a,t))^{-1}\frac{dt}{t}\end{aligned} \tag{4.6}$$

Therefore, by Eqs. 4.3, 4.4, 4.5, 4.6, we obtain Eq. 4.1 and this proves the proposition. □

Proposition 4.3 *Let $1 < p < \infty$, $(w_1, w_2) \in A_{p/s}$ and $w_2 \in A_{p/s}$ where $1 \le s < p$, and $w_1 \in A_\infty$. If S is bounded from L^{p,w_1} to L^{p,w_2}, then there exists a constant $C > 0$ such that*

$$\|[b,S]f\|_{L^{p,w_2}(B(a,r))} \le C\|b\|_* w_2(B(a,r))^{\frac{1}{p}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) w_2(B(a,t))^{-\frac{1}{p}} \|f\|_{L^{p,w_1}B(a,t)} \frac{dt}{t} \tag{4.7}$$

for any $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$ and $f \in L^{p,w_1}_{\mathrm{loc}}$.

Remark 4.4 We shall see in the proof of Theorem 1.7 that the right-hand side of Eq. 4.7 is finite if f belongs to some generalized weighted Morrey spaces under some assumptions.

Proof of Proposition 4.3 Let $a \in \mathbb{R}^n$ and $r > 0$, and we represent the function $f \in L^{p,w_1}_{\mathrm{loc}}$ as $f := f_1 + f_2$ where $f_1 := f \cdot \chi_{B(a,2r)}$. Since $[b,S]$ is bounded from L^{p,w_1} to L^{p,w_2}, then

$$\|[b,S]f_1\|_{L^{p,w_2}(B(a,r))} \le \|[b,S]f_1\|_{L^{p,w_2}} \le C\|b\|_*\|f_1\|_{L^{p,w_1}} = C\|b\|_*\|f\|_{L^{p,w_1}(B(a,2r))}$$

and Lemma 2.23 implies that

$$\|[b,S]f_1\|_{L^{p,w_2}(B(a,r))} \le C\|b\|_* w_2(B(a,r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} w_2(B(a,t))^{-\frac{1}{p}} \frac{dt}{t} \tag{4.8}$$

On the other hand, we have the estimate

$$|[b,S]f_2(x)| \le C|b(x) - b_{B(a,r)}||S(f_2)(x)| + |S((b_{B(a,r)} - b)(f_2)(x)| := CI_1(x) + CI_2(x)$$

for $x \in B(a,r)$ whenever $I_1(x)$ and $I_2(x)$ make senses. Note that $I_1(x)$ makes sense provided the right-hand side of Eq. 1.4 is finite and $I_2(x)$ makes sense provided the right-hand side of

$$\sup_{x\in B(a,r)} \left|S\left((b_{B(a,r)} - b)(f)\cdot\chi_{B(a,2r)^c}\right)(x)\right| \le C\sum_{k=1}^\infty (2^{k+1}r)^{-\frac{n}{s}} \left(\int_{B(a,2^{k+1}r)} |(b(y) - b_{B(a,r)})f(y)|^s dy\right)^{1/s} \tag{4.9}$$

is also finite. For $I_1(x)$, as in proof of the previous proposition, we obtain that the right-hand side of Eq. 1.4 is finite and

$$I_1(x) \le C|b(x) - b_{B(a,r)}| \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} w_2(B(a,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

The assumption $w_2^s \in A_{p/s}$ implies $w_2 \in A_p$, and by Lemma 2.20, it follows that

$$\|I_1\|_{L^{p,w_2}(B(a,r))} \le C\|b\|_* w_2(B(a,r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L^{p,w_1}(B(a,t))} w_2(B(a,t))^{-\frac{1}{p}} \frac{dt}{t} \tag{4.10}$$

For $I_2(x)$, we first note that Hölder's inequality implies

$$\begin{aligned}&\left(\int_{B(a,2^{k+1}r)} |(b(y) - b_{B(a,r)})f(y)|^s dy\right)^{\frac{1}{s}}\\ &\qquad \le C\|f\|_{L^{p,w_1}(B(a,2^{k+1}r))} \left\|(b - b_{B(a,r)})w_1^{-\frac{1}{p}}\right\|_{L^{ps/(p-s)}(B(a,2^{k+1}r))}.\end{aligned}$$

Thus, by the assumption that $w_1 \in A_\infty$, Lemma 2.20, and Lemma 2.23,

$$
\begin{aligned}
&\sum_{k=1}^{\infty}(2^{k+1}r)^{-\frac{n}{s}}\left(\int_{B(a,2^{k+1}r)}|(b(y)-b_{B(a,r)})f(y)|^s dy\right)^{\frac{1}{s}}\\
&\quad = C\sum_{k=1}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\frac{dt}{t^{n+1}}\|f\|_{L^{p,w_1}(B(a,2^{k+1}r))}\|(b-b_{B(a,r)})w_1^{-\frac{1}{p}}\|_{L^{\frac{ps}{p-s}}(B(a,2^{k+1}r))}\\
&\quad \le C\int_{2r}^{\infty}\|f\|_{L^{p,w_1}(B(a,t))}\|(b-b_{B(a,r)})w_1^{\frac{1}{p}}\|_{L^{\frac{ps}{p-s}}(B(a,t))}\frac{dt}{t^{n+1}}\\
&\quad \le C\int_{2r}^{\infty}\|f\|_{L^{p,w_1}(B(a,t))}\left(\int_{B(a,t)}|b(y)-b_{B(a,r)}|^{\frac{ps}{p-s}}w_1(y)^{-\frac{s}{p-s}}dy\right)^{\frac{p-s}{ps}}\frac{dt}{t^{n+1}}\\
&\quad = C\int_{2r}^{\infty}\|f\|_{L^{p,w_1}(B(a,t))}\|b\|_*\left(1+\ln\frac{t}{r}\right)\left(w_1^{-\frac{s}{p-s}}(B(a,t))\right)^{\frac{p-s}{ps}}\frac{dt}{t^{n+1}}\\
&\quad = C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}\|w_1^{-1}\|_{L^{\frac{s}{p-s}}(B(a,t))}\frac{dt}{t^{-\alpha+n+1}}\\
&\quad \le C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t}
\end{aligned}
$$

which follows that the right-hand side of Eq. 4.9 is finite. Moreover,

$$
I_2(x) \le C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t}, \quad x\in B(a,r).
$$

Hence,

$$
\|I_2\|_{L^{p,w_2}(B(a,r))} \le Cw_2(B(a,r))\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t} \tag{4.11}
$$

By combining Eqs. 4.10 and 4.11, we then obtain

$$
\|[b,S]f_2\|_{L^{p,w_2}(B(a,r))} \le C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t} \tag{4.12}
$$

Finally, Eqs. 4.8 and 4.12 yield that

$$
\|[b,S]f\|_{L^{p,w_2}(B(a,r))} \le C\|b\|_*\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L^{p,w_1}(B(a,t))}w_2(B(a,t))^{-\frac{1}{p}}\frac{dt}{t}
$$

which proves the proposition. □

The argument concerning $S(f_2)$ and $S((b_{B(a,r)}-b)f_2)$ in the following proposition may be handled in the same way in Proposition 4.3, and we omit the details here.

Proposition 4.5 *Let $(w_1,w_2)\in A_1$, $w_2\in A_1$, and $w_1\in A_\infty$. If S satisfies Eq. 1.11, then there exists a constant $C>0$ such that for any $\sigma>0$, $(a,r)\in\mathbb{R}^n\times\mathbb{R}^+$, and f,*

$$
\|[b,S](f)\|_{WL^{1,w_2}} \le C\sigma\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.
$$

Proof Let $a\in\mathbb{R}^n$ and $r>0$, and we represent the function f as $f:=f_1+f_2$ where $f_1:=f\cdot\chi_{B(a,2r)}$. We then have that

$$
\begin{aligned}
w\left(\{x\in B(a,r):|[b,S](f)(x)|>\sigma\}\right) &\le w\left(\left\{x\in B(a,r):|[b,S](f_1)(x)|>\frac{\sigma}{2}\right\}\right)+\\
&w\left(\left\{x\in B(a,r):|[b,S](f_2)(x)|>\frac{\sigma}{2}\right\}\right):=M_1+M_2
\end{aligned}
$$

Since S satisfies Eq. 1.11, by Eq. 2.1 and Lemma 2.23, we have that

$$\begin{aligned}M_1 &\le C\int_{\mathbb{R}^n}\Phi\left(\frac{2|f_1(y)|}{\sigma}\right)w_1(x)dx \le C\int_{\mathbb{R}^n}\Phi\left(\frac{2|f_1(y)|}{\sigma}\right)w_1(x)dx\\&\le C\int_{B(a,2r)}\Phi\left(\frac{2|f_1(y)|}{\sigma}\right)w_1(x)dx\\&\le Cw_2(B(a,2r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L^{1,w}(B(a,t))}w_2(B(a,t))^{-1}\frac{dt}{t}\\&\le Cw_2(B(a,r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.\end{aligned}$$

Hence,

$$\|[b,S]f_1\|_{WL^{1,w_2}(B(a,r))} \le C\sigma\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}. \tag{4.13}$$

By the sublinearity,

$$\begin{aligned}&w_2\left(\left\{x\in B(a,r): |[b,S](f_2)(x)|>\frac{\sigma}{2}\right\}\right)\\&\le w_2\left(\left\{x\in B(a,r): |b(x)-b_{B(a,r)}|\cdot|S(f_2)(x)|>\frac{\sigma}{4}\right\}\right)\\&+w_2\left(\left\{x\in B(a,r): |S((b_{B(a,r)}-b)(f_2))(x)|>\frac{\sigma}{4}\right\}\right):=M_1'+M_2'.\end{aligned}$$

Note that as in Theorem 4.1, by the fact that $t\le\Phi(t)$ and Eq. 2.2, we have

$$\begin{aligned}\frac{1}{\sigma}|S(f_2)(x)| &\le C\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B(a,r)|}\int_{2^{k+1}B(a,r)}\left|\frac{f(y)}{\sigma}\right|dy\\&\le C\int_{2r}^{\infty}\left\|\frac{f}{\sigma}\right\|_{L^{1,w}(B(a,t))}w_2(B(a,t))^{-1}\frac{dt}{t}\\&\le C\int_{2r}^{\infty}\left\|\Phi\left(\frac{f}{\sigma}\right)\right\|_{L^{1,w}(B(a,t))}w_2(B(a,t))^{-1}\frac{dt}{t}\\&\le C\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.\end{aligned}$$

For M_1', the last inequality implies that

$$\begin{aligned}M_1' &\le \frac{1}{\sigma}\int_{B(a,r)}|b(x)-b_{B(a,r)}|\cdot|S(f_2)(x)|w_2(x)dx\\&\le C\left(\int_{B(a,r)}|b(x)-b_{B(a,r)}|w_2(x)dx\right)\\&\quad\times\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}\\&\le C\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1B(a,t)}{w_2(B(a,t))}\frac{dt}{t}\end{aligned}$$

For M_2', by the assumption $(w_1, w_2) \in A_1$, we have

$$
\begin{aligned}
M_2' &\le C\frac{1}{\sigma}\int_{B(a,r)} |S((b_{B(a,r)} - b) f_2)(x)| w_2(x) dx \\
&\le w_2(B(a,r))\frac{1}{\sigma}\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B(a,r)|}\int_{2^{k+1}B(a,r)} |(b_{B(a,r)} - b(x)) f(x)| dy \\
&\le C w_2(B(a,r))\frac{1}{\sigma}\int_{2r}^{\infty} \|(b_{B(a,r)} - b) f\|_{L^{1,w_1}(B(a,t))}\|w_1^{-1}\|_{L^{\infty}(B(a,t))}\frac{dt}{t^{n+1}} \\
&\le C w_2(B(a,r))\int_{2r}^{\infty}\left\|(b_{B(a,r)} - b)\frac{f}{\sigma}\right\|_{L^{1,w_1}(B(a,t))} w_2(B(a,t))^{-1}\frac{dt}{t} \\
&\le C w_2(B(a,r))\int_{2r}^{\infty}\left\|(b_{B(a,t)} - b)\frac{f}{\sigma}\right\|_{L^{1,w_1}(B(a,t))} w_2(B(a,t))^{-1}\frac{dt}{t} \\
&\qquad + C w_2(B(a,r))\int_{2r}^{\infty}\left\|(b_{B(a,r)} - b_{B(a,t)})\frac{f}{\sigma}\right\|_{L^{1,w_1}(B(a,t))} w_2(B(a,t))^{-1}\frac{dt}{t} \\
&:= M_2'' + M_2'''
\end{aligned}
$$

By Eq. 2.1 and Lemma 2.21, we obtain

$$
\begin{aligned}
M_2'' &\le C w_2(B(a,r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))} \\
&\qquad \times \|b - b_{B(a,t)}\|_{\exp L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t} \\
&\le C\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.
\end{aligned}
$$

For M_2''', by the fact $t \le \Phi(t)$ and Lemma 2.18, we get

$$
\begin{aligned}
M_2''' &= w_2(B(a,r))\int_{2r}^{\infty}(b_{B(a,r)} - b_{B(a,t)})\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L^{1,w_1}(B(a,t))} w_2(B(a,t))^{-1}\frac{dt}{t} \\
&\le C\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(\ln\frac{t}{r}\right)\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.
\end{aligned}
$$

Hence,

$$
M_2' \le C\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right)\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.
$$

Therefore,

$$
\begin{aligned}
& w_2\left(\left\{x \in B(a,r) : |[b,S](f_2)(x)| > \frac{\sigma}{2}\right\}\right) \\
&\le C\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right)\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}
\end{aligned}
$$

and

$$
\begin{aligned}
\|[b,S](f_2)\|_{WL^{1,w_2}} &\le C\sigma\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right) \\
&\qquad \times \left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}.
\end{aligned}
\tag{4.14}
$$

By combining Eqs. 4.13 and 4.14, we obtain that

$$
\|[b,S](f)\|_{WL^{1,w_2}} \le C\sigma\|b\|_* w_2(B(a,r))\int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right)\left\|\Phi\left(\frac{|f|}{\sigma}\right)\right\|_{L\log L^{w_1}(B(a,t))}\frac{w_1 B(a,t)}{w_2(B(a,t))}\frac{dt}{t}
$$

and this proves the proposition. □

We omit the proof of Theorem 1.6 and Theorem 1.7 since those are very similar to Theorem 1.2 and Theorem 1.3. We shall prove Theorem 1.8, Theorem 1.10, Theorem 1.11, and Theorem 1.12.

Proof of Theorem 1.8 We set $h(t,r) = 1 + \ln(t/r)$ for $(t,r) \in \mathbb{R}^+ \times \mathbb{R}^+$ and

$$u_1(a,r) = \frac{w_1(B(a,r))}{w_2(B(a,r))}, \quad v_1(a,r) = \psi_1(a,r)$$
$$u_2(a,r) = 1, \qquad v_2(a,r) = \psi_2(a,r)$$

for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$. By Lemma 2.22, the following inequality

$$\frac{1}{\psi_2(a,r)} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L^{w_1}(B(a,t))} \frac{w_1 B(a,t)}{w_2(B(a,t))} \frac{dt}{t} \le C\|b\|_* \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathcal{M}_{\psi_1}^{L\log L, w_1}}$$

holds for $(a,r) \in \mathbb{R}^n \times \mathbb{R}^+$. Proposition 4.5, assumptions, and the last inequality then imply that

$$\begin{aligned}
\|[b,S]f\|_{W\mathcal{M}_{\psi_2}^{1,w_2}} &= \sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \frac{1}{w_2(B(a,r))^{\frac{1}{p}}} \|[b,S]f\|_{WL^{1,w_2}(B(a,r))} \\
&\le C\sigma\|b\|_* \sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \\
&\quad \times \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{L\log L^{w_1}(B(a,t))} \frac{w_1 B(a,t)}{w_2(B(a,t))} \frac{dt}{t} \\
&\le C\sigma\|b\|_* \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathcal{M}_{\psi_1}^{L\log L, w_1}} \sup_{(a,r)\in\mathbb{R}^n\times\mathbb{R}^+} \frac{1}{\psi_2(a,r)} \int_{r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \\
&\quad \times \inf_{t\le s<\infty} \psi_1(a,s) \frac{w_1(B(a,t))}{w_2(B(a,t))} \frac{dt}{t} \\
&\le C\sigma\|b\|_* \left\| \Phi\left(\frac{|f|}{\sigma}\right) \right\|_{\mathcal{M}_{\psi_1}^{L\log L, w_1}},
\end{aligned}$$

and this proves the theorem. □

Proof of Theorem 1.10 $f(\cdot)$ may be replaced by $f(\cdot,t)$ as well as S by T_0 in Proposition 4.1 to obtain

$$\|T_0 f(\cdot,t)\|_{\mathcal{M}_{\psi_2}^{q,w_2}} \le C\|b\|_* \|f(\cdot,t)\|_{\mathcal{M}_{\psi_1}^{p,w_1}}, \quad f \in \mathcal{M}_{\psi_1}^{p,w_1},$$

where $C > 0$ is independent of t and $1 < p < \infty$. The theorem about the boundedness of S on generalized weighted mixed-Morrey spaces follows from Definition 2.15 and Remark 2.17. □

Proof of Theorem 1.11 $f(\cdot)$ may be replaced by $f(\cdot,t)$ as well as S by T_0 in Proposition 4.3 to obtain that

$$\|[b,T_0]f(\cdot,t)\|_{\mathcal{M}_{\psi_2}^{q,w_2}} \le C\|b\|_* \|f(\cdot,t)\|_{\mathcal{M}_{\psi_1}^{p,w_1}}, \quad f \in \mathcal{M}_{\psi_1}^{p,w_1},$$

where $C > 0$ is independent of t and $1 < p < \infty$. Then, Definition 2.15 and Remark 2.17 imply the result. □

Proof of Theorem 1.12 $f(\cdot)$ may be replaced by $f(\cdot,t)$ in Theorem 4.5 to obtain

$$w_2(\{x \in \mathbb{R}^n : |[b,T_0](f)(x,t)| > \sigma\}) \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x,t)|}{\sigma}\right) w_1(x)dx,$$

where $C > 0$ is independent of t. By Definition 2.15 and Remark 2.17, we obtain the desired results about the boundedness of the commutator generated by S and b on generalized weighted mixed-Morrey spaces in the case $p = 1$. □

Applications to other operators

Fractional integrals related to operators with Gaussian kernel bounds, fractional operators with rough kernels, and fractional maximal operators with rough kernels

We provide the applications of our results related to sublinear operator generated by fractional integral (Theorem 1.2, Theorem 1.3, Theorem 1.4, and Theorem 1.5) for some other operators particularly fractional integrals and fractional maximal operators, i.e., $L^{-\alpha/2}$, $S_{\Omega,\alpha}$, and $M_{\Omega,\alpha}$. First, we recall the definition and basic results of the operators. Let L be a linear operator on L^2 which generates an analytic semigroup $\exp(-tL)$ with a kernel $p_t(x, y)$ satisfying

$$|p_t(x, y)| \leq C\frac{1}{t^{n/2}}\exp\left(-c\frac{|x-y|^2}{t}\right); \quad x, y \in \mathbb{R}^n, t > 0.$$

For $0 < \alpha < n$, the fractional power $L^{-\alpha/n}$ of the operator L is defined by

$$L^{-\alpha/2}(f)(x) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty \exp(-tL)(f)(x)\frac{dt}{t^{-\alpha/2+1}}.$$

Recall that the operator $L^{-\alpha/2}$ and its commutator are bounded on weighted Lebesgue spaces, as stated in the following theorem.

Theorem 5.1 *[2] $L^{-\alpha/2}$ and its commutator $[b, L^{-\alpha/2}]$ where $b \in BMO$ is bounded from L^{p,w^p} to L^{q,w^q} for $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$.*

Next, suppose that $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ where $n \geq 2$ equipped with the normalized Lebesgue measure $d\sigma$. Let $\Omega \in L^{s'}(\mathbb{S}^{n-1})$ be homogeneous of degree zero where $s' > 1$. For $0 < \alpha < n$, the fractional integrals with rough kernel Ω is defined by

$$S_{\Omega,\alpha}(f)(x) = \int_{\mathbb{R}^n}\frac{\Omega(y)}{|y|^{n-\alpha}}f(x-y)dy.$$

For $M_{\Omega,\alpha}$, suppose that $0 < \alpha < n$ and $\Omega \in L^{s'}(\mathbb{S}^{n-1})$ is homogeneous of degree zero where $s' > 1$. The fractional maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}(f)(x) = \sup_{r>0}\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)<r}|\Omega(x-y)f(y)|dy.$$

The boundedness properties of these operators and their commutators on weighted Lebesgue spaces are given in the following theorem.

Theorem 5.2 *[12, 13] Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $b \in BMO$. Suppose that $\Omega \in L^{s'}(\mathbb{S})^{n-1}$ and $w^s \in A_{p/s,q/s}$ where $1 \leq s < p$. Then, $S_{\Omega,\alpha}$, $M_{\Omega,\alpha}$, $[b, S_{\Omega,\alpha}]$, and $[M_{\Omega,\alpha}]$ is bounded from L^{p,w^p} to L^{q,w^q}.*

Related to inequality Eq. 1.1, we have the following lemma.

Lemma 5.3 *[57] Suppose that $0 < \alpha < n$.*

1. *$L^{-\alpha/2}$ satisfies Eq. 1.1.*
2. *If $\Omega \in L^{s'}(\mathbb{S}^{n-1})$ is homogeneous of degree zero where $1 < s' \leq \infty$, then $S_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ satisfy Eq. 1.1.*

The following theorems generalize the results in [47, 57] and others by using Theorem 5.1 and Theorem 5.2. The first theorem provides sufficient conditions for the boundedness of the sublinear operator generated by $L^{-\alpha/2}$ and its commutator within the framework of generalized weighted Morrey spaces. Similarly, the second theorem establishes the boundedness of these operators on generalized weighted mixed-Morrey spaces under specific conditions.

Theorem 5.4 *Let $0 < \alpha < n$, $1 \leq p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \leq s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $L^{-\alpha/2}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then $L^{-\alpha/2}$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$ for $1 < p < n/\alpha$ and from $\mathcal{M}_{\psi_1}^{1,w}$ to $W\mathcal{M}_{\psi_2}^{q,w_2^q}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, L^{-\alpha/2}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, L^{-\alpha/2}]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$ for $1 < p < n/\alpha$.*

Proof $L^{-\alpha/2}$ satisfies Eq. 1.1 by Lemma 5.3. Plug $L^{-\alpha/2}$ onto S_α in Theorem 1.2 and Theorem 1.3 to obtain the results. □

Theorem 5.5 *Let $0 < \alpha < n$, $1 \leq p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \leq s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that $1 < p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $L^{-\alpha/2}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then $L^{-\alpha/2}$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$ for $1 < p < n/\alpha$ and from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{1,w})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,W\mathcal{M}_{\psi_2}^{q,w^q})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, L^{-\alpha/2}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, L^{-\alpha/2}]$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$ for $1 < p < n/\alpha$.*

Proof Just plug $L^{-\alpha/2}$ onto S_α in Theorem 1.4 and Theorem 1.5. □

The previous results imply the following corollaries regarding the boundedness of the operator $L^{-\alpha/2}$ on generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces.

Corollary 5.6 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $L^{-\alpha/2}$ is bounded from $\mathcal{M}_{\psi_1}^{p,w^p}$ to $\mathcal{M}_{\psi_2}^{q,w^q}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. If $b \in BMO$, then $[b, L^{-\alpha/2}]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w^p}$ to $\mathcal{M}_{\psi_2}^{q,w^q}$.*

Corollary 5.7 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $1 \le p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $L^{-\alpha/2}$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w^q})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. If $b \in BMO$, then $[b, L^{-\alpha/2}]$ is bounded from $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w^q})$.*

Next, we consider the operator $S_{\Omega,\alpha}$ related to the kernel Ω. The subsequent two theorems provide the two-weight boundedness results for the sublinear operator generated by $S_{\Omega,\alpha}$ and its commutator on generalized weighted Morrey spaces as well as generalized weighted mixed-Morrey spaces.

Theorem 5.8 *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $S_{\Omega,\alpha}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w_2^q}, then $S_{\Omega,\alpha}$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$ for $1 < p < n/\alpha$ and from $\mathcal{M}_{\psi_1}^{1,w}$ to $W\mathcal{M}_{\psi_2}^{q,w^q}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, S_{\Omega,\alpha}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, S_{\Omega,\alpha}]$ is bounded from $\mathcal{M}_{\psi_1}^{p,w_1^p}$ to $\mathcal{M}_{\psi_2}^{q,w_2^q}$ for $1 < p < n/\alpha$.*

Proof $S_{\Omega,\alpha}$ satisfies Eq. 1.1 by Lemma 5.3 point (2). The results immediately are obtained by plugging $S_{\Omega,\alpha}$ onto S_α in Theorem 1.2 and Theorem 1.3. □

Theorem 5.9 *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, $1 \le p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $S_{\Omega,\alpha}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < \infty$ and from L^{1,w_1} to WL^{q,w^q}, then $S_{\Omega,\alpha}$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$ for $1 < p < n/\alpha$ and from $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{1,w})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,W\mathcal{M}_{\psi_2}^{q,w^q})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, S_{\Omega,\alpha}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, S_{\Omega,\alpha}]$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0,T,\mathcal{M}_{\psi_2}^{q,w_2^q})$ for $1 < p < n/\alpha$.*

Proof Just plug $S_{\Omega,\alpha}$ onto S_α in Theorem 1.4 and Theorem 1.5. □

If we assume $w = w_1 = w_2$ in the two preceding theorems, we deduce the following corollaries. These corollaries provide boundedness results for $S_{\Omega,\alpha}$ and its commutator on both generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces.

Corollary 5.10 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $S_{\Omega,\alpha}$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. If $b \in BMO$, then $[b, S_{\Omega,\alpha}]$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$.*

Corollary 5.11 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, $1 \le p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $S_{\Omega,\alpha}$ is bounded from the space $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1^p}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{q,w_2^q}_{\psi_2})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. Then, $[b, S_{\Omega,\alpha}]$ is bounded from the space $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1^p}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{q,w_2^q}_{\psi_2})$.*

Now, the two-weight boundedness results for the sublinear operator generated by $M_{\Omega,\alpha}$ and its commutator on generalized weighted Morrey and mixed-Morrey spaces are presented in the following theorems.

Theorem 5.12 *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $M_{\Omega,\alpha}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w^q}, then $M_{\Omega,\alpha}$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$ for $1 < p < n/\alpha$ and from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, M_{\Omega,\alpha}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, M_{\Omega,\alpha}]$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$ for $1 < p < n/\alpha$.*

Proof By Lemma 5.3, $M_{\Omega,\alpha}$ satisfies Eq. 1.1. We plug $M_{\Omega,\alpha}$ onto S_α in Theorem 1.2 and Theorem 1.3 to obtain the desired boundedness properties. □

Theorem 5.13 *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $1/q = 1/p - \alpha/n$, $(w_1^s, w_2^s) \in A_{p/s,q/s}$, and $w_2^s \in A_{p/s,q/s}$, where $1 \le s < p$ for $1 < p < n/\alpha$ and $s = 1$ for $p = 1$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, $1 \le p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. If $M_{\Omega,\alpha}$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$ and from L^{1,w_1} to WL^{q,w^q}, then $M_{\Omega,\alpha}$ is bounded from the space $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1^p}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{q,w_2^q}_{\psi_2})$ for $1 < p < n/\alpha$ and from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{1,w}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, W\mathcal{M}^{q,w^q}_{\psi_2})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7 and $w_1 \in A_\infty$. If $b \in BMO$ and $[b, M_{\Omega,\alpha}]$ is bounded from L^{p,w_1^p} to L^{q,w_2^q} for $1 < p < n/\alpha$, then $[b, M_{\Omega,\alpha}]$ is bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1^p}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{q,w_2^q}_{\psi_2})$ for $1 < p < n/\alpha$.*

Proof Put $M_{\Omega,\alpha} = S_\alpha$ in Theorem 1.4 and Theorem 1.5. □

Finally, by taking $w = w_1 = w_2$ in the two preceding theorems, we obtain the following corollaries.

Corollary 5.14 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $M_{\Omega,\alpha}$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. If $b \in BMO$, then $[b, M_{\Omega,\alpha}]$ is bounded from $\mathcal{M}^{p,w_1^p}_{\psi_1}$ to $\mathcal{M}^{q,w_2^q}_{\psi_2}$.*

Corollary 5.15 *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^s \in A_{p/s,q/s}$, where $1 \le s < p$. Suppose that $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, $1 \le p_0 < \infty$ and $\psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$.*

1. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.6. Then, $M_{\Omega,\alpha}$ is bounded from the space $\mathcal{M}_{\psi}^{p_0,w}(0, T, \mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0, T, \mathcal{M}_{\psi_2}^{q,w_2^q})$.*
2. *Suppose that the positive functions ψ_1 and ψ_2 satisfy Eq. 1.7. If $b \in BMO$, then $[b, M_{\Omega,\alpha}]$ is bounded from $\mathcal{M}_{\psi}^{p_0,w}(0, T, \mathcal{M}_{\psi_1}^{p,w_1^p})$ to $\mathcal{M}_{\psi}^{p_0,w}(0, T, \mathcal{M}_{\psi_2}^{q,w_2^q})$.*

Singular integral operators

Let Ω be as in the "Fractional integrals related to operators with Gaussian kernel bounds, fractional operators with rough kernels, and fractional maximal operators with rough kernels" section. Suppose that S_Ω is a sublinear operator such that for any $f \in L^1$ with compact support and $x \notin \text{supp}\,(f)$

$$|S_\Omega(f)(x)| \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy. \tag{5.1}$$

In fact, S_Ω which may be viewed as the generalization of Calderón-Zygmund operator (see [7, 19] for more details about Calderón-Zygmund operator) satisfies Eq. 1.1 as in the following lemma.

Lemma 5.16 *S_Ω satisfies Eq. 1.1 for $\Omega \in L^{s'}(\mathbb{S}^{n-1})$ where $1 < s' \le \infty$.*

Proof By Hölder's's inequality, we deduce

$$\begin{aligned}
|S_\Omega f \cdot \chi_{B(a,2r)^c})(x)| &\le C \int_{B(a,2r)^c} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\
&= C \sum_{k=1}^{\infty} \int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\
&\le \sum_{k=1}^{\infty} \int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\
&\le C \sum_{k=1}^{\infty} \left(\int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} |\Omega(x-y)|^{s'} dy \right)^{\frac{1}{s'}} \left(\int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} \frac{|f(y)|^s}{|x-y|^{ns}} dy \right)^{\frac{1}{s}}.
\end{aligned}$$

We note that

$$\begin{aligned}
\left(\int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} |\Omega(x-y)|^{s'} dy \right)^{\frac{1}{s'}} &\le C \|\Omega\|_{L^{s'}(\mathbb{S}^{n-1})} |B(0, 2^{k+1}r + |x-a|)|^{\frac{1}{s'}}, \\
&\le C \|\Omega\|_{L^{s'}(\mathbb{S}^{n-1})} |B(a, 2^{k+1}r)|^{\frac{1}{s'}}
\end{aligned}$$

where $C > 0$ is independent of x, k, and a (see [25] for more details). We also note that $|x-y| \sim |y-a|$ for $x \in B(a, r)$ and $y \in B(a, 2r)^c$. Hence,

$$\left(\int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} \frac{|f(y)|^s}{|x-y|^{ns}} dy \right)^{\frac{1}{s}} \le C \frac{1}{|B(a, 2^{k+1}r)|} \left(\int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} |f(y)|^s dy \right)^{\frac{1}{s}}.$$

Consequently,

$$|S_\Omega(f \cdot \chi_{B(a,2r)^c})(x)| \le C \sum_{k=1}^{\infty} \left(\frac{1}{|B(a, 2^{k+1}r)|} \int_{B(a,2^{k+1}r)\setminus B(a,2^k r)} |f(y)|^s dy \right)^{\frac{1}{s}}$$

which implies that S_Ω satisfies Eq. 1.1, and the theorem is proved. □

Based on the preceding lemma, we present the following theorem concerning the boundedness of the sublinear operator with a rough kernel and its commutator on generalized weighted Morrey and mixed-Morrey spaces with different weights. Furthermore, we obtain the weak-type $(1, 1)$ estimates for S_α and the weak-type $(L \log L, 1)$ estimates for $[b, S_\alpha]$. The following theorem generalizes several known results in the literature, including those found in [25, 29, 30, 37, 46, 47].

Theorem 5.17 *Let $1 \leq p < \infty$, $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, (w_1, w_2), $w_2 \in A_{p/s}$ where $1 \leq s < p$ for $p > 1$ and $s = 1$ for $p = 1$, and S_Ω satisfies Eq. 5.1.*

1. *Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.8. If S_Ω is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$ and bounded from L^{1,w_1} to WL^{1,w_2}, then S_Ω is bounded from $\mathcal{M}^{p,w_1}_{\psi_1}$ to $\mathcal{M}^{p,w_2}_{\psi_2}$ for $1 < p < \infty$ and bounded from $\mathcal{M}^{1,w_1}_{\psi_1}$ to $W\mathcal{M}^{1,w_2}_{\psi_2}$.*
2. *Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.9 and $w_1 \in A_\infty$. If $[b, S_\Omega]$ is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$, then $[b, S_\Omega]$ is bounded from $\mathcal{M}^{p,w_1}_{\psi_1}$ to $\mathcal{M}^{p,w_2}_{\psi_2}$ for $1 < p < \infty$.*
3. *Suppose that the two positive functions ψ_1 and ψ_2 on $\mathbb{R}^n \times \mathbb{R}^+$ satisfy Eq. 1.10 and $w_1 \in A_\infty$. If S_Ω satisfies Eq. 1.11, then there exists a constant $C > 0$ such that*

$$\|[b, S_\Omega](f)\|_{W\mathcal{M}^{1,w_2}_{\psi_2}} \leq C\|b\|_* \sup_{\sigma>0} \sigma \left\| \Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{L\log L, w_1}_{\psi_1}}.$$

Proof S_Ω satisfies Eq. 1.1 by Lemma 5.16. Plug S_Ω onto S in Theorem 1.6, Theorem 1.7, and Theorem 1.8 to obtain the desired results. □

Theorem 5.18 *Let ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$, $0 < p_0 < \infty$, $1 \leq p < \infty$, $\Omega \in L^{s'}(\mathbb{S}^{n-1})$, and (w_1, w_2), $w_2 \in A_{p/s}$, where $1 \leq s < p$ for $p > 1$ and $s = 1$ for $p = 1$. Suppose that $1 \leq p_0 < \infty$, $\psi : \mathbb{R}^n \times \mathbb{R}$, and S_Ω satisfies Eq. 5.1.*

1. *Suppose that the two functions ψ_1 and ψ_2 satisfy Eq. 1.8. If S_Ω is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$ and bounded from L^{1,w_1} to WL^{1,w_2}, then S_Ω is bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1}_{\psi_1})$ to the space $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_2}_{\psi_2})$ for $1 < p < \infty$ and bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{1,w_1}_{\psi_1})$ to the space $\mathcal{M}^{p_0,w}_{\psi}(0, T, W\mathcal{M}^{1,w_2}_{\psi_2})$.*
2. *Suppose that the two functions ψ_1 and ψ_2 satisfy Eq. 1.9 and $w_1 \in A_\infty$. If S_Ω is bounded from L^{p,w_1} to L^{p,w_2} for $1 < p < \infty$, then $[b, S_\Omega]$ is bounded from $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_1}_{\psi_1})$ to $\mathcal{M}^{p_0,w}_{\psi}(0, T, \mathcal{M}^{p,w_2}_{\psi_2})$ for $1 < p < \infty$.*
3. *Suppose that the two functions ψ_1 and ψ_2 satisfy Eq. 1.10 and $w_1 \in A_\infty$. If S_Ω satisfies Eq. 1.11, then there exists a constant $C > 0$ such that for any suitable function f,*

$$\|[b, S_\Omega](f)\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,W\mathcal{M}^{1,w_2}_{\psi_2})} \leq C\|b\|_* \sup_{\sigma>0} \sigma \left\| \Phi\left(\frac{|f|}{\sigma}\right)\right\|_{\mathcal{M}^{p_0,w}_{\psi}(0,T,\mathcal{M}^{L\log L, w_1}_{\psi_1})}.$$

Proof Just plug S_Ω onto S in Theorem 1.10, Theorem 1.11, and Theorem 1.12. □

Applications to partial differential equations

In this section, we investigate the regularity properties of the solution of partial differential equations, i.e., elliptic partial differential equation and parabolic partial differential equation. We use the boundedness properties obtained in the previous sections particularly fractional integral operator I_α and singular integral operators K. Here, we take $w_1 = w_2 = w$ the weight on $\mathbb{R}^n$. We also assume that $1 \leq p_0 < \infty$, w_0 a weight on Ω, and ψ a positive function on $\mathbb{R}^n \times \mathbb{R}^+$.

Elliptic partial differential equations

Throughout this subsection, let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$ where $n \geq 2$. We always assume that $1 \leq p < \infty$ and write $1/q = 1/p - 2/n$ and $1/q_0 = 1/p - 1/n$.

For $k = 1, 2$, we denote $W^{1,k}(\Omega)$ by the Sobolev spaces and under the Sobolev norm, the closure of $C_0^\infty(\Omega)$ in $W^{1,k}(\Omega)$ is denoted by $W_0^{1,k}(\Omega)$. Moreover, we denote $H^{-1}(\Omega)$ by the dual space of $W_0^{1,2}(\Omega)$. We shall investigate the following Dirichlet problem

$$Lu = f \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega, \quad u \in W_o^{1,2}(\Omega) \tag{6.1}$$

where L is the divergent elliptic operator. The function f is belonging to generalized weighted Morrey spaces $\mathcal{M}^{p,w^p}_{\psi_1}$. The operator L is defined by

$$Lu = -\sum_{i,j=1}^{\infty} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)$$

where $u \in W_0^{1,2}(\Omega)$, $a_{ij} = a_{ji} \in L^\infty(\Omega)$ for $i, j, \in \{1, \ldots, n\}$, and there exists $\Lambda > 0$ for which

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

for $x \in \Omega$ a.e. Moreover, we assume that the coefficients of L satisfy the Dini- continuous condition.

Theorem 6.1 *[20] There exists a unique function $K : \Omega \times \Omega \to [0, \infty]$ such that*

$$G(\cdot, y) \in W^{1,2}(\Omega \backslash B(y, r)) \cap W_0^{1,1}(\Omega), \quad (y, r) \in \Omega \times \mathbb{R}^+,$$

and for $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial G(x, y)}{\partial x_i} \frac{\partial (x)}{\partial x_j} dx = \phi(y).$$

Furthermore,

$$G(x, y) \leq \frac{C}{|x - y|^{n-2}}, \quad |\nabla_x G(x, y)| \leq \frac{C}{|x - y|^{n-1}}, \quad x, y \in \Omega, x \neq y.$$

The function G in the theorem is then called the Green function for the operator L and the domain Ω. For $f \in \mathcal{M}_\psi^{p,w}(\Omega) \cap H^{-1}(\Omega)$, we define

$$u(x) = \int_\Omega G(x, y) f(y) dy, \quad x \in \Omega. \tag{6.2}$$

Lemma 6.2 *[54] The weak derivative of u is given by*

$$\frac{\partial u(x)}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\int_\Omega G(x, y) f(y) dy \right) = \int_\Omega \frac{\partial G(x, y)}{\partial x_i} f(y) dy.$$

The subsequent theorem details the boundedness properties of the operator given in Eq. 6.2 on generalized weighted Morrey spaces, encompassing both strong-type (p, q) estimates for $1 < p < \infty$ and weak-type $(1, 1)$ estimates.

Theorem 6.3 *For $1 < p < \infty$, $w \in A_{p,q}$, and the pair of functions (ψ_1, ψ_2) satisfying Eq. 1.6, there exists a positive constant C such that*

$$\|u\|_{\mathcal{M}_{\psi_2}^{q,w^q}(\Omega)} \leq C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}(\Omega)}, \quad \|u\|_{W\mathcal{M}_{\psi_2}^{q,w^q}(\Omega)} \leq C\|f\|_{\mathcal{M}_{\psi_1}^{1,w}(\Omega)}.$$

Proof One may apply Theorem 1.2 and then use the definition of u as in Eq. 6.2 to get the desired inequality. □

Meanwhile, the next theorem presents the boundedness properties of the operator given in Eq. 6.2 in the setting of generalized weighted mixed-Morrey spaces.

Theorem 6.4 *For $1 < p < \infty$, $w \in A_{p,q}$, and the pair of functions (ψ_1, ψ_2) satisfying Eq. 1.6 there exists a positive constant C such that*

$$\|u\|_{\mathcal{M}_\psi^{p_0,w_0}(0,T\mathcal{M}_{\psi_2}^{q,w^q}(\Omega))} \leq C\|f\|_{\mathcal{M}_\psi^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{p,w^p}(\Omega))},$$

$$\|u\|_{\mathcal{M}_\psi^{p_0,w_0}(0,T,W\mathcal{M}_{\psi_2}^{q,w^q}(\Omega))} \leq C\|f\|_{\mathcal{M}_\psi^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{1,w}(\Omega))}.$$

The regularity properties of the solution to the system Eq. 6.1 are established in our next theorem, which relies on the two preceding theorems, Theorem 4 in [54], and the results from [10, 11].

Theorem 6.5 *Under the same assumptions of Theorem 6.4, u is the unique weak solution of Dirichlet problem Eq. 6.1 and $u \in \mathcal{M}_{\psi_2}^{q,w^q}(\Omega) \cap \mathcal{M}_\psi^{p_0,w_0}(0, T, \mathcal{M}_{\psi_2}^{q,w^q}(\Omega))$.*

Turning our attention to the gradient of u, our next results reveal its boundedness across both generalized weighted Morrey spaces and generalized weighted mixed-Morrey spaces.

Theorem 6.6 *For $1 < p < \infty$, $w \in A_{p,q_0}$, and the pair of functions (ψ_1, ψ_2) satisfying Eq. 1.6 where q is replaced by q_0, there exists a constant C such that*

$$\|\nabla u\|_{\mathcal{M}_{\psi_2}^{q_0,w^{q_0}}(\Omega)} \leq C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}(\Omega)}, \quad \|\nabla u\|_{W\mathcal{M}_{\psi_2}^{q_0,w^{q_0}}(\Omega)} \leq C\|f\|_{\mathcal{M}_{\psi_1}^{1,w}(\Omega)}.$$

Proof Note that by Theorem 6.1 and Lemma 6.2, we have

$$|\nabla u| = \left(\sum_{i=1}^{n}\left|\int_{\Omega}\frac{\partial G(x,y)}{\partial x_i}f(y)dy\right|^2\right)^{\frac{1}{2}} \le \sqrt{n}\int_{\Omega}|\nabla_x G(x,y)||f(y)|dy \le \sqrt{n}\int_{\Omega}\frac{1}{|x-y|^{n-1}}|f(y)|dy,$$

for $x \in \Omega$. Hence, by Theorem 1.2, we obtain the desired inequality. □

Theorem 6.7 *For $1 < p < \infty$, $w \in A_{p,q_0}$, and the pair of functions (ψ_1, ψ_2) satisfying Eq. 1.6 where q is replaced by q_0, there exists a constant C such that*

$$\|\nabla u\|_{\mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_2}^{q_0,w^{q_0}}(\Omega))} \le C\|f\|_{\mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{p,w^p}(\Omega))},$$
$$\|\nabla u\|_{\mathcal{M}_{\psi}^{p_0,w_0}(0,T,W\mathcal{M}_{\psi_2}^{q_0,w^{q_0}}(\Omega))} \le C\|f\|_{\mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{1,w}(\Omega))}.$$

Parabolic partial differential equations

First, we recall the Sarason class $VMO = VMO(\mathbb{R}^n)$. The BMO function f is in VMO if $\lim_{r\to 0^+}\sup_{\rho\ge r}\frac{1}{B_\rho}\int_{B_\rho}|f(y)-f_{B_\rho}|dy = 0$ where B_ρ is a ball with radius ρ. We also recall that the singular integral operator K and its commutator as in the "Applications to other operators" section are bounded on $L^{p,w}$ (see [26, 51] for more details). Through this subsection, we assume that $1 < p < \infty$ and the pairing functions (ψ_1, ψ_2) satisfy Eq. 1.9. Next, we recall the definition of singular integral operator related to variable Calderòn-Zymgund kernel.

Definition 6.8 Suppose that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is measurable function. K is called a variable Calderón-Zygmund kernel provided the following condition holds.

1. $K(x,\cdot)$ is a Calderón-Zygmund kernel for almost all $x \in \mathbb{R}^n$, i.e.:
   (a) $K(x,\cdot) \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$,
   (b) $K(x,\mu y) = \mu^{-n}K(x,\mu)$ for $\mu > 0$,
   (c) $\int_{\mathbb{S}^{n-1}} K(x,y)d\sigma_y = 0$ and $\int_{\mathbb{S}^{n-1}}|K(x,y)|d\sigma_y$.
2. $\max_{|\beta|\le 2n}\left\|D_y^{\beta}K\right\|_{L^{\infty}(\mathbb{R}^n\times\mathbb{S}^{n-1})}$ is finite.

We define the singular integral $K(f)$ and the commutator $[a,K](f)$ by

$$K(f)(x) = P.V.\int_{\mathbb{R}^n}K(x,x-y)f(y)dy$$
$$[a,K](f)(x) = P.V.\int_{\mathbb{R}^n}K(x,x-y)[a(x)-a(y)]f(y)dy.$$

Note that K satisfies Eq. 1.1 when $f \in L_1(\mathbb{R}^n)$ with compact support where Ω constant a.e. as in [27]. Hence, we have the following corollary about the boundedness of K and its commutator on generalized weighted Morrey spaces generalizing the results in [27, 46, 47, 56] and on generalized weighted mixed-Morrey spaces which generalize the results in [46].

Corollary 6.9 *Let ψ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and $1 < p_0 < \infty$. Then, K and $[b,K]$ are bounded from $\mathcal{M}_{\psi_1}^{p,w}$ to $\mathcal{M}_{\psi_2}^{p,w}$ and $\mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{p,w})$ to $\mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_2}^{p,w})$.*

Let $n \ge 3$ and $Q_T = \Omega \times (0,T)$ be a cylinder of $\mathbb{R}^{n+1}$ of base $\Omega \subset \mathbb{R}^n$. In the sequel, we write $x' = (x,t) = (x_1,x_2,\cdots,x_n,t) \in \mathbb{R}^{n+1}$ a generic point in Q_T, $f \in \mathcal{M}_{\psi_1}^{p,w}(\Omega)$ or $f \in \mathcal{M}_{\psi}^{p_0,w_0}(0,T,\mathcal{M}_{\psi_1}^{p,w}(\Omega))$ where $1 \le p_0, p < \infty$. We also write

$$Lu = u_t - \sum_{i,j=1}^{n}a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}$$

where $a_{ij}(x,t) = a_{ji}(x,t)$ *a.e.* $x \in Q_T$; $a_{ij}(x,t) \in VMO(Q_T) \cap L^{\infty}(Q_T)$ for $i,j = 1,\cdots,n$, and

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2; \quad \textit{a.e. in } Q_T, \xi \in \mathbb{R}^n,$$

Suppose the equation $Lu(x,t) = f(x,t)$. The strong solution of the equation is a function $u \in \mathcal{M}^{p,w}_{\psi_1}(\Omega)$ if $f \in u \in \mathcal{M}^{p,w}_{\psi_1}(\Omega)$ and $u \in \mathcal{M}^{p_0,w_0}_{\psi}(0,T,\mathcal{M}^{p,w}_{\psi_1}(\Omega))$ if $f \in u \in \mathcal{M}^{p_0,w_0}_{\psi}(0,T,\mathcal{M}^{p,w}_{\psi_1}(\Omega))$ with all its weak derivatives $D_{x'_i}u$, $D_{x'_i x'_j}u$, and D_t satisfying the equation as in [46]. The local representation formula (see [4] for more details about the formula and notations) for $D_{x'_i,x'_j}u$ where $u \in C_t = \{v \in C_0^{\infty}(\mathbb{R}^{n+1} \cap \{t \geq 0\}) : v(x,0) = 0\}$ is given by

$$\begin{aligned} D_{x'_i,x'_j} &= \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x,x-y)Lu(y)dy \\ &+ \lim_{\varepsilon\to 0}\int_{\rho(x-y)>\varepsilon} \Gamma_{ij}(x,x-y)\sum_{h,k=1}^{n}[a_{hk}(y)-a_{hk}(x)]D_{x'_h,x'_k}u(y)dy \\ &+ Lu(x)\int_{\Sigma}\Gamma_j(x,y)v_i d\sigma(y) \\ &= \mathcal{R}_{ij}Lu(x) + \sum_{h,k=1}^{n}[a_{hk},\mathcal{R}_{ij}](D_{x'_h,x'_k}u)(x) + Lu(x)\int_{\Sigma}\Gamma_j(x,y)v_i d\sigma(y), \end{aligned} \tag{6.3}$$

for $x \in \operatorname{supp}(f)$ where v_i is the $i-$th component of the outer normal to Σ. Here $\Gamma_{ij}(x,\xi) = D_{\xi_i,\xi_j}\Gamma(x,\xi)$ and Γ_{ij} are Calderón-Zygmund kernel as in Definition 6.8. This implies that $\mathcal{R}_{ij}$ are singular integrals as in the "Applications to other operators" section. Moreover, we may observe that

$$u_t = Lu + \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Therefore, the regularity properties of the solution to the parabolic partial differential equations are established in the next theorems by utilizing Corollary 6.9, the local representation 6.3, and the boundedness of both K and $[b,K]$ on $L^{p,w}$ whenever $b \in VMO$.

Theorem 6.10 *Let $n \geq 3$, $a_{ij} \in VMO(Q_T) \cap L^{\infty}(Q_T)$, $B_r \subseteq \Omega$ a ball in $\mathbb{R}^n$. For every u with compact support in $B \times (0,T)$, the solution of $Lu = f$ such that $D_{x'_i,x'_j}u \in \mathcal{M}^{p,w}_{\psi_1}(B_r)$ for $i,j = 1,\cdots,n$, there exists r_0 such that if $r < r_0$, then*

$$\|D_{x'_i,x'_j}u\|_{\mathcal{M}^{p,w}_{\psi_2}(B_r)} \leq C\|Lu\|_{\mathcal{M}^{p,w}_{\psi_1}(B_r)}, \quad i,j = 1,\cdots,n$$

and

$$\|u_t\|_{\mathcal{M}^{p,w}_{\psi_2}(B_r)} \leq C\|Lu\|_{\mathcal{M}^{p,w}_{\psi_1}(B_r)}.$$

Theorem 6.11 *Let $n \geq 3$, $a_{ij} \in VMO(Q_T) \cap L^{\infty}(Q_T)$, and $B_r \subseteq \Omega$ a ball in $\mathbb{R}^n$. For every u with compact support in $B \times (0,T)$, the solution of $Lu = f$ such that $D_{x'_i,x'_j}u \in \mathcal{M}^{p_0}_{\psi}(0,T\mathcal{M}^{p,w}_{\psi_1}(B_r))$ for $i,j = 1,\cdots,n$, there exists r_0 such that if $r < r_0$, then*

$$\|D_{x'_i,x'_j}u\|_{\mathcal{M}^{p_0}_{\psi}(0,T\mathcal{M}^{p,w}_{\psi_2}(B_r))} \leq \|Lu\|_{\mathcal{M}^{p_0}_{\psi}(0,T\mathcal{M}^{p,w}_{\psi_1}(B_r))}, \quad i,j = 1,\cdots,n,$$

and

$$\|u_t\|_{\mathcal{M}^{p_0}_{\psi}(0,T\mathcal{M}^{p,w}_{\psi_2}(B_r))} \leq \|Lu\|_{\mathcal{M}^{p_0}_{\psi}(0,T\mathcal{M}^{p,w}_{\psi_1}(B_r))}.$$

Remark 6.12 Note that the last theorem generalizes Theorem 5.1 in [46].

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