The Journal of Analysis
https://doi.org/10.1007/s41478-026-01132-7

ORIGINAL RESEARCH PAPER

Some geometric constants for mixed morrey spaces and their discrete versions

Ifronika[1] · **Hendra Gunawan**[2] · **Janny Lindiarni**[2]

Received: 17 February 2026 / Accepted: 6 June 2026

Abstract

In this paper, we compute several geometric constants for mixed Morrey spaces and mixed Morrey double-sequence spaces, specifically generalized Von Neumann-Jordan constant, modified Von Neumann-Jordan constants, and Zbáganu constant. For the calculation of these constants, we construct two functions of unit norm so that the upper bound of these constants is attained. In addition, we use an inequality involving these constants and Von Neumann-Jordan constant. Our results can be seen as some extensions of the previous results about geometric constants for Morrey spaces and discrete Morrey spaces.

Keywords Von Neumann-Jordan constant · Generalized Von Neumann-Jordan constant · Modified Von Neumann-Jordan constants · Zbáganu constant · Mixed Morrey spaces · Mixed Morrey double-sequence spaces

H. Gunawan and J. Lindiarni have contributed equally to this work.

Communicated by S Ponnusamy.

✉ Hendra Gunawan
hgunawan@itb.ac.id

Ifronika
ifronika@itb.ac.id

Janny Lindiarni
janny@itb.ac.id

[1] Doctoral Program of Mathematics, Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, Jl. Ganesha 10, Bandung 40132, Indonesia

[2] Department of Mathematics, Faculty of Mathematics and Natural Sciences, Bandung Institute of Technology, Jl. Ganesha 10, Bandung 40132, Indonesia

Published online: 06 July 2026

1 Introduction

In 1961, Benedek and Panzone studied Lebesgue spaces with mixed norm [1]. Similarly, Morrey spaces with mixed norm and their application to partial differential equations are investigated by Ragusa and Scapellato in 2017 [2]. Other studies about Morrey spaces, mixed Morrey spaces and their variations can be found in [3–8].

For $1 \le p \le q < \infty$ and $1 \le r \le s < \infty$, we define the *mixed Morrey space* $\mathcal{M}^p_q(\mathcal{M}^r_s) = \mathcal{M}^p_q(\mathcal{M}^r_s)(\mathbb{R}^d \times \mathbb{R}^d)$ to be the set of all measurable functions f on $\mathbb{R}^d \times \mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}^p_q(\mathcal{M}^r_s)} := \|\|f(x,y)\|_{\mathcal{M}^r_s,x}\|_{\mathcal{M}^p_q,y} < \infty.$$

Here, $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ denotes the classical Morrey space, which is the set of all measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p dy\right)^{\frac{1}{p}} < \infty.$$

Although Morrey spaces are known and there are many references about these spaces, the study of their discrete versions is relatively new (see [9, 10]). Let $1 \le p \le q < \infty$. The discrete Morrey spaces or Morrey sequence spaces $\ell^p_q = \ell^p_q(\mathbb{Z}^d)$ is defined to be the set of all sequences $x : \mathbb{Z}^d \to \mathbb{R}$ for which

$$\|\{x_j\}\|_{\ell^p_q} := \sup_{m\in\mathbb{Z}^d, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{j\in S_{m,N}} |x_j|^p\right)^{\frac{1}{p}} < \infty,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $S_{m,N} = \{j \in \mathbb{Z}^d : \|j-m\|_\infty \le N\}$.

Inspired by the definition of mixed Morrey spaces, the authors in [11] introduced the mixed Morrey double-sequence space $\ell^p_q(\ell^r_s) = \ell^p_q(\ell^r_s)(\mathbb{Z}\times\mathbb{Z})$, where $1 \le p \le q < \infty$ and $1 \le r \le s < \infty$, to be the set all double-sequences $\{x_{ij}\} = \{x_{ij}\}_{i,j\in\mathbb{Z}}$ for which

$$\|\{x_{ij}\}\|_{\ell^p_q(\ell^r_s)} := \|\{\|\{x_{ij}\}\|_{\ell^r_s,i}\}\|_{\ell^p_q,j} < \infty.$$

In this paper, we extend the domain of these sequences spaces to $\mathbb{Z}^d \times \mathbb{Z}^d$, so that we have $\ell^p_q(\ell^r_s)(\mathbb{Z}^d \times \mathbb{Z}^d)$.

The aim of this paper is to extend the previous results about the Von Neumann-Jordan constants for mixed Morrey spaces [12] and mixed Morrey double-sequence spaces [11]. In particular, we compute generalized Von Neumann-Jordan constant, modified Von Neumann-Jordan constant, generalized modified Von Neumann-Jordan Constant, and Zbăganu constant for these spaces. For the mixed Morrey double-sequence spaces, we also compute the Von Neumann-Jordan constant and the James constant.

For a Banach space $(X, \|\cdot\|_X)$, we denote by $S_X := \{x \in X : \|x\|_X = 1\}$ the unit sphere in X. The Von Neumann-Jordan constant and the James constant for X are given respectively by

$$C_{NJ}(X) := \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2(\|x\|_X^2 + \|y\|_X^2)} : x, y \in X\backslash\{0\}\right\}$$

and

$$C_J(X) := \sup\{\min(||x+y||_X, ||x-y||_X) : x, y \in S_X\}.$$

Meanwhile the definitions of the other constants are given below. For $1 \le t < \infty$, the generalized Von Neumann-Jordan Constant $C_{NJ}^t(X)$ [13] is defined by

$$C_{NJ}^t(X) := \sup\left\{\frac{\|x+y\|_X^t + \|x-y\|_X^t}{2^{t-1}(\|x\|_X^t + \|y\|_X^t)} : x, y \in X\backslash\{0\}\right\}.$$

By the convexity of the function $f(x) = x^t$ for $t \ge 1$ and the triangle inequality, it follows that $C_{NJ}^t(X) \le 2$. Moreover, for $t = 2$, we have $C_{NJ}^t(X) = C_{NJ}(X)$.

The modified Von Neumann-Jordan constant $C'_{NJ}(X)$ is defined by

$$C'_{NJ}(X) := \sup\left\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{4} : x, y \in S_X\right\}.$$

This constant is introduced by Alonso et al. [14]. One can see that, the definition of $C_{NJ}(X)$ and $C'_{NJ}(X)$ imply $1 \le C'_{NJ}(X) \le C_{NJ}(X)$.

As a combination of $C'_{NJ}(X)$ and $C_{NJ}^t(X)$, for $1 \le t < \infty$, the generalized modified Von Neumann-Jordan constant $\bar{C}_{NJ}^t(X)$ [15], is defined by

$$\bar{C}_{NJ}^t(X) := \sup\left\{\frac{\|x+y\|_X^t + \|x-y\|_X^t}{2^t} : x, y \in S_X\right\}.$$

For $t = 1$, we have the following constant

$$A_2(X) = \sup\left\{\frac{\|x+y\|_X + \|x-y\|_X}{2} : x, y \in S_X\right\},$$

as investigated in [16].

Zbáganu [17] introduced a different constant than Von Neumann-Jordan constant, namely Zbáganu constant, which is defined by

$$C_Z(X) := \sup\left\{\frac{\|x+y\|_X\|x-y\|_X}{\|x\|_X^2 + \|y\|_X^2} : x, y \in X \setminus \{0\}\right\},$$

One can observe that $1 \le C_Z(X) \le C_{NJ}(X) \le 2$.

In this paper, we will calculate the values of the above constants for mixed Morrey spaces and mixed Morrey double-sequence spaces. Specifically, we prove that for $X = \mathcal{M}_q^p(\mathcal{M}_s^r)$ and $X = \ell_q^p(\ell_s^r)$, the value of the above geometric constants are equal to the value of $C_{NJ}(X)$.

2 Results

Our first result for mixed Morrey spaces $\mathcal{M}_q^p(\mathcal{M}_s^r)$ is presented in the following theorem.

Theorem 1 *For $1 \leq p < q < \infty$, $1 \leq r < s < \infty$ and $1 \leq t < \infty$, we have*

$$C_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) = C'_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) = \bar{C}_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) = C_Z(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2.$$

Proof As in [12], let $I_\alpha^\beta = [\beta - \alpha, \beta)$ denote an interval of length α ending at the point β and define $(I_\alpha^\beta)^d$ as the Cartesian product

$$(I_\alpha^\beta)^d = I_a^\beta \times \cdots \times I_a^\beta,$$

where $\alpha, \beta \in \mathbb{R}$. We then define the following functions

$$f(x) = \chi_{(I_1^1)^d}(x) + \chi_{I_1^{n+1} \times (I_1^1)^{d-1}}(x), \quad x \in \mathbb{R}^d$$

and

$$g(x) = \chi_{(I_1^1)^d}(x) - \chi_{I_1^{n+1} \times (I_1^1)^{d-1}}(x), \quad x \in \mathbb{R}^d.$$

Here, χ_A denotes the characteristic function of A, for every $A \subseteq \mathbb{R}^d$, and $n \geq \max\left\{2^{\frac{q}{d(q-p)}} - 1, 2^{\frac{s}{d(s-r)}} - 1\right\}$.

Note that

$$\|f\|_{\mathcal{M}_q^p} = \|f\|_{\mathcal{M}_s^r} = \|g\|_{\mathcal{M}_q^p} = \|g\|_{\mathcal{M}_s^r} = 1.$$

Let us now define

$$\overline{f}(x, y) = f(x)f(y) \text{ and } \overline{g}(x, y) = g(x)g(y), \quad x, y \in \mathbb{R}^d,$$

so that

$$\|\overline{f}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|f\|_{\mathcal{M}_q^p}\|f\|_{\mathcal{M}_s^r} = 1$$

and

$$\|\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|g\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_s^r} = 1.$$

One may observe that

$$\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|\overline{f}-\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 2.$$

See [12] for the details. It follows that

$$\begin{aligned} C_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) &\geq \frac{\|\overline{f}+\overline{g}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{f}-\overline{g}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{2^{t-1}\left(\|\overline{f}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{g}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)}\right)} \\ &= \frac{2^t+2^t}{2^{t-1}(1+1)} = 2. \end{aligned}$$

Since $C_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) \leq 2$, we get $C_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2$.

Next, for the modified Von Neumann-Jordan constant, we have

$$\begin{aligned} C'_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) &\geq \frac{\|\overline{f}+\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{f}-\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{4} \\ &= \frac{2^2+2^2}{4} = 2. \end{aligned}$$

We know that $C'_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) \leq C_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) \leq 2$, thus we can conclude that $C'_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2$.

We also obtain the result for the generalized modified Von Neumann–Jordan constant as follows,

$$\begin{aligned} \bar{C}_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) &\geq \frac{\|\overline{f}+\overline{g}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{f}-\overline{g}\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{2^t} \\ &= \frac{2^t+2^t}{2^t} = 2. \end{aligned}$$

Therefore, we conclude that $\bar{C}_{NJ}^t(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2$.

Finally, for the Zbáganu constant, we have

$$\begin{aligned} C_Z(\mathcal{M}_q^p(\mathcal{M}_s^r)) &\geq \frac{\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}\|\overline{f}-\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{\|\overline{f}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)}} \\ &= \frac{2\cdot 2}{2} = 2. \end{aligned}$$

Since $C_Z(\mathcal{M}_q^p(\mathcal{M}_s^r)) \leq C_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2$, we obtain $C_Z(\mathcal{M}_q^p(\mathcal{M}_s^r)) = 2$.

We now state the results for mixed Morrey double-sequence spaces. As the reader might observe, the key idea of the proof of the previous theorem is to find two elements whose norms are equal to one and the norms of their difference and sum are equal to two. For mixed Morrey double-sequence spaces, we find that the same idea works and we state the result as in the following theorem.

Theorem 2 *Let $1 \leq p < q < \infty$ and $1 \leq r < s < \infty$. Then, there exist $u, v \in \ell^p_q(\ell^r_s)$ with $\|u\|_{\ell^p_q(\ell^r_s)} = \|v\|_{\ell^p_q(\ell^r_s)} = 1$ such that*

$$\|u+v\|_{\ell^p_q(\ell^r_s)} = \|u-v\|_{\ell^p_q(\ell^r_s)} = 2.$$

Proof The constructions of *u* and *v* are adapted from [18] and [11]. Following a similar approach, we construct two sequences of unit norm so that the norm of the sums and difference of these sequences are equal to two. Since $q > p$ and $s > r$, we can choose $N_0 \in \mathbb{N}$ such that

$$N_0 \geq \max\left\{2^{\frac{q}{d(q-p)}} - 1, 2^{\frac{s}{d(s-r)}} - 1\right\}.$$

For a nonnegative integer $n \in \mathbb{N}$, we define the mappings $x, y : \mathbb{Z}^d \to \mathbb{R}$ representing the sequences $x = \{x_i\}$ and $y = \{y_i\}$, by

$$x_i := \begin{cases} 1, & i = (0,0,\dots,0), (n,0,\dots,0), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$y_i := \begin{cases} 1, & i = (0,0,\dots,0) \\ -1, & i = (n,0,\dots,0) \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have

$$\begin{aligned} \|x\|_{\ell^p_q} = \|\{x_i\}\|_{\ell^p_q} &= \sup_{m\in\mathbb{Z}^d, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{i\in S_{m,N}} |x_i|^p\right)^{\frac{1}{p}} \\ &= \max\left(1, (n+1)^{d\left(\frac{1}{q}-\frac{1}{p}\right)} 2^{\frac{1}{p}}\right). \end{aligned}$$

If $n \geq N_0$, we obtain $(n+1)^{d\left(\frac{1}{q}-\frac{1}{p}\right)} 2^{\frac{1}{p}} < 1$. Hence $\|x\|_{\ell^p_q} = 1$. By a similar argument, one also obtains $\|x\|_{\ell^r_s} = \|y\|_{\ell^p_q} = \|y\|_{\ell^r_s} = 1$.

Next, we introduce the double sequences $u = \{u_{ij}\}$ and $v = \{v_{ij}\}$, defined by

$$u_{ij} = x_i x_j \quad \text{and} \quad v_{ij} = y_i y_j \quad i, j \in \mathbb{Z}^d. \tag{1}$$

It follows that,

$$\|u\|_{\ell^p_q(\ell^r_q)} = \|x\|_{\ell^r_s}\|x\|_{\ell^p_q} = 1 \quad \text{and} \quad \|v\|_{\ell^p_q(\ell^r_q)} = \|y\|_{\ell^r_s}\|y\|_{\ell^p_q} = 1. \tag{2}$$

We now proceed to compute $\|u \pm v\|_{\ell^p_q(\ell^r_s)}$. From the definitions of x and y, we have

$$x_i + y_i = \begin{cases} 2, & i = (0,0,\ldots,0) \\ 0, & i = (n,0,\ldots,0) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x_i - y_i = \begin{cases} 0, & i = (0,0,\ldots,0) \\ 2, & i = (n,0,\ldots,0) \\ 0, & \text{otherwise.} \end{cases}$$

Let us first consider $\|u + v\|_{\ell^p_q(\ell^r_s)}$. Let $i \in \mathbb{Z}^d$. For $j = (0,0,\ldots,0)$, we have

$$u + v = x_i x_j + y_i y_j = x_i \cdot 1 + y_i \cdot 1 = x_i + y_i,$$

so that

$$\|u + v\|_{\ell^r_s, i} = \|x + y\|_{\ell^r_s} = 2.$$

For $j = (n,0,\ldots,0)$, we obtain

$$u + v = x_i x_j + y_i y_j = x_i \cdot 1 + y_i \cdot (-1) = x_i - y_i,$$

hence

$$\|u + v\|_{\ell^r_s, i} = \|x - y\|_{\ell^r_s} = 2.$$

For $j \notin \{(0,0,\ldots,0),(n,0,\ldots,0)\}$, we have

$$u + v = x_i x_j + y_i y_j = x_i \cdot 0 + y_i \cdot 0 = 0,$$

so that

$$\|u + v\|_{\ell^r_s, i} = 0.$$

These calculations can be summarized as

$$\begin{aligned} \|u + v\|_{\ell^r_s, i} &= \begin{cases} 2, & j \in \{(0,0,\ldots,0),(n,0,\ldots,0)\} \\ 0, & j \notin \{(0,0,\ldots,0),(n,0,\ldots,0)\} \end{cases} \\ &= 2x_j. \end{aligned}$$

Consequently,

$$\|u + v\|_{\ell^p_q(\ell^r_s)} = \left\| \|u + v\|_{\ell^r_s, i} \right\|_{\ell^p_q, j} = \|2x\|_{\ell^p_q} = 2. \tag{3}$$

By the same argument, one finds that

$$\|u - v\|_{\ell^p_q(\ell^r_s)} = 2. \tag{4}$$

□

As a direct consequence of Theorem 2, we obtain the following corollary.

Corollary 3 *If* $1 \leq p < q < \infty$ *and* $1 \leq r < s < \infty$*, then*

$$C_{NJ}(\ell^p_q(\ell^r_s)) = C_J(\ell^p_q(\ell^r_s)) = 2.$$

Proof We first consider the Von Neumann–Jordan constant. Observe that, $C_{NJ}(\ell^p_q(\ell^r_s)) \leq 2$. Thus, it remains to prove $C_{NJ}(\ell^p_q(\ell^r_s)) \geq 2$. Let *u* and *v* be the sequences defined in the proof of Theorem 2. Then, we obtain

$$C_{NJ}(\ell^p_q(\ell^r_s)) \geq \frac{\|u+v\|^2_{\ell^p_q(\ell^r_s)} + \|u-v\|^2_{\ell^p_q(\ell^r_s)}}{2(\|u\|^2_{\ell^p_q(\ell^r_s)} + \|v\|^2_{\ell^p_q(\ell^r_s)})} = \frac{2^2+2^2}{2(1+1)} = 2.$$

For the James constant, let the sequences *u* and *v* be defined by (1). As a consequence of Theorem 2, we have

$$\begin{aligned} C_J(\ell^p_q(\ell^r_s)) &= \sup\Big\{ \min\Big\{\|u+v\|_{\ell^p_q(\ell^r_s)}, \|u-v\|_{\ell^p_q(\ell^r_s)}\Big\} \\ &\qquad : \|u\|_{\ell^p_q(\ell^r_s)} = \|v\|_{\ell^p_q(\ell^r_s)} = 1\Big\} \\ &= 2, \end{aligned}$$

which completes the proof.

The above corollary shows that both Von Neumann–Jordan and James constants for mixed Morrey double-sequence spaces attain their maximum value. The following theorem shows that the same is true for the other geometric constants.

Theorem 4 *For* $1 \leq p < q < \infty$ *and* $1 \leq r < s < \infty$*, we have* $C^t_{NJ}(\ell^p_q(\ell^r_s)) = C'_{NJ}(\ell^p_q(\ell^r_s)) = \bar{C}^t_{NJ}(\ell^p_q(\ell^r_s)) = C_Z(\ell^p_q(\ell^r_s)) = 2$.

Proof We shall now calculate the generalized Von Neumann-Jordan. Since $C^t_{NJ}(\ell^p_q(\ell^r_s)) \leq 2$, it remains to prove that $C^t_{NJ}(\ell^p_q(\ell^r_s)) \geq 2$. According to the Eqs. (2–4), we obtain

$$\begin{aligned} C^t_{NJ}(\ell^p_q(\ell^r_s)) &\geq \frac{\|u+v\|^t_{\ell^p_q(\ell^r_s)} + \|u-v\|^t_{\ell^p_q(\ell^r_s)}}{2^{t-1}\left(\|u\|^t_{\ell^p_q(\ell^r_s)} + \|v\|^t_{\ell^p_q(\ell^r_s)}\right)} \\ &= \frac{2^t+2^t}{2^{t-1}(1+1)} = 2. \end{aligned}$$

Hence, we conclude that $C^t_{NJ}(\ell^p_q(\ell^r_s)) = 2$.

Next, we consider the modified Von Neumann-Jordan constant. We have

$$C'_{NJ}(\ell_q^p(\ell_s^r)) \geq \frac{\|u+v\|^2_{\ell_q^p(\ell_s^r)} + \|u-v\|^2_{\ell_q^p(\ell_s^r)}}{4} = \frac{2^2+2^2}{4} = 2.$$

Because $C'_{NJ}(\ell_q^p(\ell_s^r)) \leq C_{NJ}(\ell_q^p(\ell_s^r)) \leq 2$, we have $C'_{NJ}(\ell_q^p(\ell_s^r)) = 2$.

Accordingly, we obtain the following result for the generalized modified Von Neumann–Jordan constant,

$$\bar{C}^t_{NJ}(\ell_q^p(\ell_s^r)) \geq \frac{\|u+v\|^t_{\ell_q^p(\ell_s^r)} + \|u-v\|^t_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{2^t} = \frac{2^t+2^t}{2^t} = 2.$$

Consequently, we obtain $\bar{C}^t_{NJ}(\ell_q^p(\ell_s^r)) = 2$.

Finally, for the Zbáganu constant, we have

$$C_Z(\ell_q^p(\ell_s^r)) \geq \frac{\|u+v\|_{\ell_q^p(\ell_s^r)}\|u-v\|_{\ell_q^p(\ell_s^r)}}{\|u\|^2_{\ell_q^p(\ell_s^r)} + \|g\|^2_{\ell_q^p(\ell_s^r)}} = \frac{2\cdot 2}{2} = 2.$$

Since, $C_Z(\ell_q^p(\ell_s^r)) \leq C_{NJ}(\ell_q^p(\ell_s^r)) = 2$, it follows that $C_Z(\ell_q^p(\ell_s^r)) = 2$.

Acknowledgements We would like to thank the referees for their useful comments and suggestions on the earlier version of this paper.

Data availability No additional data available.

Declarations

Conflict of interest The authors declare that they do not have conflict of interest.

References

1. Benedek, A., and R. Panzone. 1961. The space L^p, with mixed norm. *Duke Math J* 28 (3): 301–324.
2. Ragusa, M. A., and A. Scapellato. 2017. Mixed Morrey spaces and their applications to partial differential equations. *Nonlinear Anal: Theory, Methods & Appl* 151:51–65.
3. Aykol, C., J. J. Hasanov, and A. Serbetci. 2025. A Characterization of Riesz Potential and Its Commutator in Local Complementary Generalized Orlicz-Morrey Spaces. *J Funct Spaces* 2025 (1): 2528537.
4. Kokilashvili, V., A. Meskhi, and M. A. Ragusa. 2019. Weighted Extrapolation in Grand Morrey Spaces and Applications to Partial Differential Equations. *Rendiconti LINCEI – Matematica e Applicazioni* 30 (1): 67–92.
5. Nogayama, T. 2019. Mixed Morrey spaces. *Positivity* 23:961–1000.

6. Tarigan, M. O. I. P., R. Gunadi, D. I. Hakim, and Ifronika. 2023. Inclusion of mixed Morrey spaces. *Adv Oper Theory* 8 (2): 16.
7. Wu, J. L., Y. D. Sun, and Y. P. Chang. 2025. Some characterizations of p-adic mixed central Campanato spaces via commutator of p-adic Hardy type operator. *Filomat* 39 (12): 3963–3976.
8. Wu, S., D. Yang, and W. Yuan. 2022. Interpolations of mixed-norm function spaces. *Bull Malaysian Math Sci Soc* 45 (1): 153–175.
9. Gunawan, H., E. Kikianty, and C. Schwanke. 2018. Discrete Morrey spaces and their inclusion properties. *Mathematische Nachrichten* 291 (8–9): 1283–1296.
10. Haroske, D. D., and L. Skrzypczak. 2020. Morrey sequence spaces: Pitt's theorem and compact embeddings. *Constructive Approximation* 51 (3): 505–535.
11. Gunawan, H., D. I. Hakim, Ifronika, and O. Neswan. 2024. Inclusion and geometric properties of mixed Morrey double-sequence spaces. *Bull Malaysian Math Sci Soc* 47 (123): 1–12.
12. Ifronika, Gunawan H., and J. Lindiarni. 2026. Geometric constants for mixed Morrey spaces. *Czechoslovak Math J* 76 (2): 711–722.
13. Cui, Y., W. Huang, H. Hudzik, and R. Kaczmarek. 2015. Generalized Von Neumann-Jordan constant and its relationship to the fixed point property. *Fixed Point Theory Appl* 40:1–11.
14. Alonso, J., P. Martin, and P. L. Papini. 2008. Wheeling around Von Neumann-Jordan constant in Banach spaces. *Studia Math* 188:135–150.
15. Yang, C., and F. Wang. 2017. An extension of a simple inequality between Von Neumann-Jordan and James constants in Banach spaces. *Acta Mathematica Sinica, English Series* 9:1287–1296.
16. Baronti, M., E. Casini, and P. L. Papini. 2000. Triangles inscribed in a semicircle, in Minkowski planes, and in normed spaces. *J Math Anal Appl* 252:124–146.
17. Zbăganu, G. 2001. An inequality of M. Radulescu and S. Radulescu which characterizes inner product spaces. *Revue Roumaine de Mathématiques Pures et Appliquées* 47:253–257.
18. Gunawan, H., E. Kikianty, Y. Sawano, and C. Schwanke. 2019. Three geometric constants for Morrey spaces. *Bull Korean Math Soc* 56 (6): 1569–1575.