GEOMETRIC CONSTANTS FOR MIXED MORREY SPACES

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Received December 2, 2025. Published online March 17, 2026.

Abstract. We compute several geometric constants for mixed Morrey spaces, namely von Neumann-Jordan constant, James constant, and Dunkl-Williams constant. Through the values of the constants that we obtain, we know that mixed Morrey spaces are not uniformly convex nor uniformly nonsquare. Our result extends previous results on geometric constants for discrete Morrey spaces.

Keywords: mixed Morrey spaces; von Neumann-Jordan constant; James constant; Dunkl-Williams constant; uniformly convex; uniformly nonsquare

MSC 2020: 42B35, 46B20

1. Introduction and main results

Let $\mathcal{Q}$ denote the family of all cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes. For $f \in L^p_{\rm loc}(\mathbb{R}^d)$, $1 \leqslant p \leqslant q < \infty$, and $Q \in \mathcal{Q}$, we write

$$\mathfrak{m}_{p,q}(f;Q) = |Q|^{1/q-1/p}\biggl(\int_Q |f(x)|^p \,{\rm d}x\biggr)^{1/p},$$

where $|Q|$ denotes the volume of Q. (Note that if $\ell(Q)$ denotes the side length of Q, then we have $|Q| = [\ell(Q)]^d$.) The *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ consists of all measurable functions $f \in L^p_{\rm loc}(\mathbb{R}^d)$ for which

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q\in\mathcal{Q}} \mathfrak{m}_{p,q}(f;Q) < \infty.$$

See [1], [14], [15] for further reading on Morrey spaces.

Among various extensions of Morrey spaces, Ragusa and Scapellato in [13] studied mixed Morrey spaces and their applications to partial differential equations. Further results on mixed Morrey spaces can be found in [12], [17], [18].

DOI: 10.21136/CMJ.2026.0457-25

As in [17], for $1 \leqslant p \leqslant q < \infty$ and $1 \leqslant r \leqslant s < \infty$, we define the *mixed Morrey space* $\mathcal{M}_q^p(\mathcal{M}_s^r) = \mathcal{M}_q^p(\mathcal{M}_s^r)(\mathbb{R}^d \times \mathbb{R}^d)$ to be the set of all measurable functions f on $\mathbb{R}^d \times \mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} := \|\|f(x,y)\|_{\mathcal{M}_s^r,x}\|_{\mathcal{M}_q^p,y} < \infty.$$

The following lemma will be useful in establishing our main results.

Lemma 1.1. *Let $1 \leqslant p \leqslant q < \infty$ and $1 \leqslant r \leqslant s < \infty$. Suppose that $g \in \mathcal{M}_s^r$ and $h \in \mathcal{M}_q^p$. If $f(x,y) = g(x)h(y)$ for $x, y \in \mathbb{R}^d$, then $f \in \mathcal{M}_q^p(\mathcal{M}_s^r)$ with*

$$\|f\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|g\|_{\mathcal{M}_s^r}\|h\|_{\mathcal{M}_q^p}.$$

Proof. By definition of the mixed Morrey space $\mathcal{M}_q^p(\mathcal{M}_s^r)$, we have

$$\begin{aligned}\|f\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} &= \|\|f(x,y)\|_{\mathcal{M}_s^r,x}\|_{\mathcal{M}_q^p,y} = \|\|g(x)h(y)\|_{\mathcal{M}_s^r,x}\|_{\mathcal{M}_q^p,y} \\ &= \|\|g(x)\|_{\mathcal{M}_s^r,x}\ h(y)\|_{\mathcal{M}_q^p,y} = \|g\|_{\mathcal{M}_s^r}\|h\|_{\mathcal{M}_q^p},\end{aligned}$$

as desired. □

In this paper, we shall be interested in computing three geometric constants for mixed Morrey spaces, namely von Neumann-Jordan constant $C_{NJ}(X)$ (see [11]), James constant $C_J(X)$ (see [6]), and Dunkl-Williams constant $C_{DW}(X)$ (see [10]), where $X = \mathcal{M}_q^p(\mathcal{M}_s^r)$. For a Banach space $(X, \|\cdot\|_X)$ in general, the three constants are defined respectively by

$$(1.1)\quad C_{NJ}(X) := \sup\biggl\{\frac{\|x+y\|_X^2 + \|x-y\|_X^2}{2(\|x\|_X^2 + \|y\|_X^2)}\colon\ x, y \in X \setminus \{0\}\biggr\},$$

$$(1.2)\quad C_J(X) := \sup\{\min(||x+y||_X, ||x-y||_X)\colon\ x, y \in X,\ \|x\|_X = \|y\|_X = 1\},$$

and

$$(1.3)\quad C_{DW}(X) := \sup\biggl\{\frac{\|x\|_X + \|y\|_X}{\|x-y\|_X}\biggl\|\frac{x}{\|x\|_X} - \frac{y}{\|y\|_X}\biggr\|_X\colon\ x, y \in X \setminus \{0\},\ x \neq y\biggr\}.$$

Let us recall some basic properties of the three constants. For every Banach space X, we have $1 \leqslant C_{NJ}(X) \leqslant 2$. Moreover, the lower bound of this inequality is attained if and only if X is a Hilbert space, see [11]. For James constant, the inequality $\sqrt{2} \leqslant C_J(X) \leqslant 2$ holds for every Banach space X and its minimum value is obtained if (but not only if) X is a Hilbert space, see [2], [6]. For Dunkl-Williams constant, we have $2 \leqslant C_{DW}(X) \leqslant 4$, where the minimum value is attained if and only if X is a Hilbert space, see [5], [10]. Recent results on geometric constants for mixed Morrey double-sequence spaces $\ell^p_q(\ell^r_s)$ can be found in [7]. For related results on 'continuous' and discrete Morrey spaces, see [8].

Our main result is the following theorem, which we shall prove in the next section.

Theorem 1.2. *For $1 \leqslant p < q < \infty$ and $1 \leqslant r < s < \infty$, we have $C_{NJ}(\mathcal{M}^p_q(\mathcal{M}^r_s)) = C_J(\mathcal{M}^p_q(\mathcal{M}^r_s)) = 2$ and $C_{DW}(\mathcal{M}^p_q(\mathcal{M}^r_s)) = 4$.*

As a consequence of the von Neumann-Jordan constant that we obtain, we have some information about the nonsquareness of mixed Morrey spaces. A Banach space X is called *uniformly nonsquare* if there exists a $\delta > 0$ such that for every $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $\|\frac{1}{2}(x-y)\|_X \geqslant 1-\delta$, we have $\|\frac{1}{2}(x+y)\|_X \leqslant 1-\delta$, see [9].

Theorem 1.3 ([16])**.** *A Banach space X is uniformly nonsquare if and only if $C_{NJ}(X) < 2$.*

Based on Theorems 1.2 and 1.3, we have the following corollary.

Corollary 1.4. *For $1 \leqslant p < q < \infty$ and $1 \leqslant r < s < \infty$, the mixed Morrey space $\mathcal{M}^p_q(\mathcal{M}^r_s)$ is not uniformly nonsquare.*

Recall that a Banach space $(X, \|\cdot\|_X)$ is said to be *nonsquare* if there are no $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ such that $\|x+y\|_X = \|x-y\|_X = 2$, see [4]. As we shall see later, we can actually conclude from the proof of Theorem 1.2 that, for $1 \leqslant p < q < \infty$ and $1 \leqslant r < s < \infty$, the mixed Morrey space $\mathcal{M}^p_q(\mathcal{M}^r_s)$ is not nonsquare.

In addition, we also have some information about the convexity properties of mixed Morrey spaces. A normed space X is called *strictly convex* if for every $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $x \neq y$, we have $\|\frac{1}{2}(x+y)\|_X < 1$. Moreover, a normed space X is called *uniformly convex* if for every $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1)$ such that for every $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ and $\|x-y\|_X \geqslant \varepsilon$, we have $\|\frac{1}{2}(x+y)\|_X \leqslant 1-\delta$, see [3].

Clearly a uniformly convex space is strictly convex. Our results tell us that mixed Morrey spaces are not uniformly convex because they are not strictly convex.

Corollary 1.5. *For $1 \leqslant p < q < \infty$ and $1 \leqslant r < s < \infty$, the mixed Morrey space $\mathcal{M}^p_q(\mathcal{M}^r_s)$ is not strictly convex. Consequently, $\mathcal{M}^p_q(\mathcal{M}^r_s)$ is not uniformly convex.*

2. Proof of Theorem 1.2

In order to reveal how we compute the constants in detail, we shall deal first with the case $d = 2$. The ideas will be copied later for the general case where $d \geqslant 1$.

2.1. The case $d = 2$. We start by evaluating the von Neumann-Jordan constant. Observe that $C_{NJ}(\mathcal{M}^p_q(\mathcal{M}^r_s)) \leqslant 2$. Thus, it remains to show that $C_{NJ}(\mathcal{M}^p_q(\mathcal{M}^r_s)) \geqslant 2$.

Let N_0 be a positive integer such that

$$N_0 \geqslant \max\{2^{q/(2(q-p))} - 1, 2^{s/(2(s-r))} - 1\}.$$

Let $I^\beta_\alpha = [\beta - \alpha, \beta)$ denote an interval of length α ending at the point β, where $\alpha, \beta \in \mathbb{R}$. For $n \geqslant N_0$, define

$$f(u) = \chi_{I^1_1 \times I^1_1}(u) + \chi_{I^{n+1}_1 \times I^1_1}(u), \quad u \in \mathbb{R}^2,$$

and

$$g(u) = \chi_{I^1_1 \times I^1_1}(u) - \chi_{I^{n+1}_1 \times I^1_1}(u), \quad u \in \mathbb{R}^2.$$

We shall compute $\|f\|_{\mathcal{M}^p_q}$ by considering three types of cubes whose intersections with the support of f are nonempty.

Case 1: For Q with $\ell(Q) \leqslant 1$ such that $Q \cap I^1_1 \times I^1_1 \neq \emptyset$, the cube Q which gives the maximum value of $\mathfrak{m}_{p,q}(f; Q)$ is $Q = I^1_1 \times I^1_1$. In this case,

$$\mathfrak{m}_{p,q}(f; Q) = 1.$$

Case 2: For Q with $\ell(Q) \leqslant 1$ such that $Q \cap I^{n+1}_1 \times I^1_1 \neq \emptyset$, the cube Q which gives the maximum value of $\mathfrak{m}_{p,q}(f; Q)$ is $Q = I^{n+1}_1 \times I^1_1$. In this case,

$$\mathfrak{m}_{p,q}(f; Q) = 1.$$

Case 3: For Q such that $Q \cap (I^1_1 \cup I^{n+1}_1) \times I^1_1 \neq \emptyset$, the largest value of $\mathfrak{m}_{p,q}(f; Q)$ is attained if $Q = I^{n+1}_{n+1} \times I^{n+1+\alpha}_{n+1}$, $-n \leqslant \alpha \leqslant 0$. For such Q we have

$$\begin{aligned}
\mathfrak{m}_{p,q}(f; Q) &= |Q|^{1/q-1/p} \biggl(\int_Q |f(u)|^p \,\mathrm{d}u\biggr)^{1/p} \\
&= |Q|^{1/q-1/p} \biggl(\int_{Q \cap I^1_1 \times I^1_1} 1^p \,\mathrm{d}u + \int_{Q \cap I^{n+1}_1 \times I^1_1} 1^p \,\mathrm{d}u\biggr)^{1/p} \\
&= ((n+1)^2)^{1/q-1/p} 2^{1/p}.
\end{aligned}$$

From the three cases we see that

$$\|f\|_{\mathcal{M}_q^p} = \max\{1, ((n+1)^2)^{1/q-1/p} 2^{1/p}\}.$$

Thus, since $n \geqslant N_0$, we obtain $\|f\|_{\mathcal{M}_q^p} = 1$.

Using the same argument, we get $\|f\|_{\mathcal{M}_s^r} = 1$, $\|g\|_{\mathcal{M}_q^p} = 1$, and $\|g\|_{\mathcal{M}_s^r} = 1$.

Now define

$$\overline{f}(u,v) = f(u)f(v) \quad \text{and} \quad \overline{g}(u,v) = g(u)g(v), \quad u,v \in \mathbb{R}^2. \tag{2.1}$$

By Lemma 1.1, we have

$$\|\overline{f}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|f\|_{\mathcal{M}_q^p}\|f\|_{\mathcal{M}_s^r} = 1 \tag{2.2}$$

and

$$\|\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|g\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_s^r} = 1. \tag{2.3}$$

We shall now compute $\|\overline{f} \pm \overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}$. First, let $u \in \mathbb{R}^2$. For $v \in I_1^1 \times I_1^1$ we have

$$(\overline{f} + \overline{g})(u,v) = f(u)\cdot 1 + g(u)\cdot 1 = f(u) + g(u).$$

Therefore,

$$\|\overline{f} + \overline{g}\|_{\mathcal{M}_s^r,u} = \|f+g\|_{\mathcal{M}_s^r} = \|2\chi_{I_1^1 \times I_1^1}\|_{\mathcal{M}_s^r} = 2.$$

Next, for $v \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1$ we get

$$(\overline{f} + \overline{g})(u,v) = 0,$$

so that

$$\|\overline{f} + \overline{g}\|_{\mathcal{M}_s^r,u} = 0.$$

For $v \in I_1^{n+1} \times I_1^1$ we compute

$$(\overline{f} + \overline{g})(u,v) = f(u)\cdot 1 + g(u)(-1) = f(u) - g(u),$$

which gives

$$\|\overline{f} + \overline{g}\|_{\mathcal{M}_s^r,u} = \|f-g\|_{\mathcal{M}_s^r} = \|2\chi_{I_1^{n+1} \times I_1^1}\|_{\mathcal{M}_s^r} = 2.$$

These cases can be summarized as

$$\begin{aligned}\|\overline{f} + \overline{g}\|_{\mathcal{M}_s^r,u} &= \begin{cases} 2, & v \in I_1^1 \times I_1^1, \\ 2, & v \in I_1^{n+1} \times I_1^1, \\ 0, & v \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1, \end{cases} \\ &= 2\chi_{I_1^1 \times I_1^1}(v) + 2\chi_{I_1^{n+1} \times I_1^1}(v) = 2f(v).\end{aligned}$$

Hence, we have

$$\|\overline{f}+\overline{g}\|_{\mathcal{M}^p_q(\mathcal{M}^r_s)} = \|\|(\overline{f}+\overline{g})(u,v)\|_{\mathcal{M}^r_s,u}\|_{\mathcal{M}^p_q,v} = \|2f\|_{\mathcal{M}^p_q} = 2. \tag{2.4}$$

A similar analysis holds for $\overline{f}-\overline{g}$. Let $u \in \mathbb{R}^2$. For $v \in I_1^1 \times I_1^1$ we have

$$(\overline{f}-\overline{g})(u,v) = f(u)\cdot 1 - g(u)\cdot 1 = f(u) - g(u),$$

and so

$$\|\overline{f}-\overline{g}\|_{\mathcal{M}^r_s,u} = \|f-g\|_{\mathcal{M}^r_s} = \|2\chi_{I_1^{n+1}\times I_1^1}\|_{\mathcal{M}^r_s} = 2.$$

For $v \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1$ we have

$$(\overline{f}-\overline{g})(u,v) = 0,$$

and so

$$\|\overline{f}-\overline{g}\|_{\mathcal{M}^r_s,u} = 0.$$

For $v \in I_1^{n+1} \times I_1^1$ we have

$$(\overline{f}-\overline{g})(u,v) = f(u)\cdot 1 - g(u)(-1) = f(u) + g(u).$$

Therefore,

$$\|\overline{f}-\overline{g}\|_{\mathcal{M}^r_s,u} = \|f+g\|_{\mathcal{M}^r_s} = \|2\chi_{I_1^1\times I_1^1}\|_{\mathcal{M}^r_s} = 2.$$

The above calculations can be summarized as

$$\begin{aligned}\|\overline{f}-\overline{g}\|_{\mathcal{M}^r_s,u} &= \begin{cases} 2, & v \in I_1^1 \times I_1^1, \\ 2, & v \in I_1^{n+1} \times I_1^1, \\ 0, & v \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1, \end{cases} \\ &= 2\chi_{I_1^1\times I_1^1}(v) + 2\chi_{I_1^{n+1}\times I_1^1}(v) = 2f(v).\end{aligned}$$

Thus, we have

$$\|\overline{f}-\overline{g}\|_{\mathcal{M}^p_q(\mathcal{M}^r_s)} = \|\|(\overline{f}-\overline{g})(u,v)\|_{\mathcal{M}^r_s,u}\|_{\mathcal{M}^p_q,v} = \|2f\|_{\mathcal{M}^p_q} = 2. \tag{2.5}$$

By combining (2.2)–(2.5), we obtain that

$$C_{NJ}(\mathcal{M}^p_q(\mathcal{M}^r_s)) \geqslant \frac{\|\overline{f}+\overline{g}\|^2_{\mathcal{M}^p_q(\mathcal{M}^r_s)} + \|\overline{f}-\overline{g}\|^2_{\mathcal{M}^p_q(\mathcal{M}^r_s)}}{2(\|\overline{f}\|^2_{\mathcal{M}^p_q(\mathcal{M}^r_s)} + \|\overline{g}\|^2_{\mathcal{M}^p_q(\mathcal{M}^r_s)})} = \frac{2^2+2^2}{2(1+1)} = 2,$$

as desired.

For the James constant, let $\overline{f}$ and $\overline{g}$ be functions as in (2.1). According to the equations (2.4) and (2.5), we have

$$\begin{aligned} &C_J(\mathcal{M}_q^p(\mathcal{M}_s^r)) \\ &\quad= \sup\{\min\{\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}, \|\overline{f}-\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}\}\colon\ \|\overline{f}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 1\} \\ &\quad= 2. \end{aligned}$$

Finally, for the Dunkl-Williams constant, let $\overline{f}$ and $\overline{g}$ be functions defined by (2.1). For $\delta > 0$, consider

$$w = (1+\delta)h + (1-\delta)k,$$

where $h = \overline{f}+\overline{g}$ and $k = \overline{f}-\overline{g}$. We keep in mind that $\|h\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|k\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 2$.

We shall prove that $\|w\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 2+2\delta$. To do so, observe that

$$w = (1+\delta)h + (1-\delta)k = (1+\delta)(\overline{f}+\overline{g}) + (1-\delta)(\overline{f}-\overline{g}) = 2\overline{f} + 2\delta\overline{g},$$

whence

$$w(u,v) = 2f(u)f(v) + 2\delta g(u)g(v).$$

Let $u \in \mathbb{R}^2$. For $v \in I_1^1 \times I_1^1$ we have

$$\begin{aligned} w(u,v) &= 2f(u)\cdot 1 + 2\delta g(u)\cdot 1 = 2f(u) + 2\delta g(u) \\ &= 2(\chi_{I_1^1\times I_1^1}(u) + \chi_{I_1^{n+1}\times I_1^1}(u)) + 2\delta(\chi_{I_1^1\times I_1^1}(u) - \chi_{I_1^{n+1}\times I_1^1}(u)) \\ &= (2+2\delta)\chi_{I_1^1\times I_1^1}(u) + (2-2\delta)\chi_{I_1^{n+1}\times I_1^1}(u) \\ &= \begin{cases} 2+2\delta, & u \in I_1^1 \times I_1^1, \\ 2-2\delta, & u \in I_1^{n+1} \times I_1^1, \\ 0, & u \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1. \end{cases} \end{aligned}$$

Hence, we have

$$\|w(u,v)\|_{\mathcal{M}_s^r,u} = 2+2\delta.$$

For $v \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1$ we have $w(u,v) = 0$, so that

$$\|w(u,v)\|_{\mathcal{M}_s^r,u} = 0.$$

For $v \in I_1^{n+1} \times I_1^1$ we have

$$\begin{aligned} w(u,v) &= 2f(u)\cdot 1 + 2\delta g(u)\cdot(-1) = 2f(u) - 2\delta g(u) \\ &= 2(\chi_{I_1^1\times I_1^1}(u) + \chi_{I_1^{n+1}\times I_1^1}(u)) - 2\delta(\chi_{I_1^1\times I_1^1}(u) - \chi_{I_1^{n+1}\times I_1^1}(u)) \\ &= (2-2\delta)\chi_{I_1^1\times I_1^1}(u) + (2+2\delta)\chi_{I_1^{n+1}\times I_1^1}(u) \\ &= \begin{cases} 2-2\delta, & u \in I_1^1 \times I_1^1, \\ 2+2\delta, & u \in I_1^{n+1} \times I_1^1, \\ 0, & u \in \mathbb{R}^2 \setminus (I_1^1 \cup I_1^{n+1}) \times I_1^1, \end{cases} \end{aligned}$$

for which we have

$$\|w(u,v)\|_{\mathcal{M}^r_s,u} = 2+2\delta.$$

We conclude that

$$\begin{aligned}\|w(u,v)\|_{\mathcal{M}^r_s,u} &= \begin{cases} 2+2\delta, & v \in I^1_1 \times I^1_1,\\ 2+2\delta, & v \in I^{n+1}_1 \times I^1_1,\\ 0, & v \in \mathbb{R}^2 \setminus (I^1_1 \cup I^{n+1}_1) \times I^1_1,\end{cases}\\ &= (2+2\delta)\chi_{I^1_1\times I^1_1}(v) + (2+2\delta)\chi_{I^{n+1}_1\times I^1_1}(v) = (2+2\delta)f(v).\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\|w\|_{\mathcal{M}^p_q(\mathcal{M}^r_s)} &= \|\|w(u,v)\|_{\mathcal{M}^r_s,u}\|_{\mathcal{M}^p_q,v} = \|(2+2\delta)f\|_{\mathcal{M}^p_q}\\ &= (2+2\delta)\|f\|_{\mathcal{M}^p_q} = 2+2\delta.\end{aligned}$$

Noting that $h+k-w = \delta(k-h)$, we compute

$$\begin{aligned}\frac{\|h+k\|_X + \|w\|_X}{\delta\|k-h\|_X}\left\|\frac{h+k}{\|h+k\|_X} - \frac{w}{\|w\|_X}\right\|_X &= \frac{2+2+2\delta}{\delta\cdot 2}\left\|\frac{h+k}{2} - \frac{w}{2+\delta}\right\|_X\\ &= \frac{2+\delta}{\delta}\left\|\frac{(1+\delta)(h+k)-w}{2+2\delta}\right\|_X\\ &= \frac{2+\delta}{\delta(2+2\delta)}\|2\delta k\|_X = \frac{2+\delta}{1+\delta}\|k\|_X\\ &= \frac{4+2\delta}{1+\delta},\end{aligned}$$

where $X = \mathcal{M}^p_q(\mathcal{M}^r_s)$. Taking $\delta \to 0^+$, we obtain $C_{DW}(\mathcal{M}^p_q(\mathcal{M}^r_s)) = 4$.

2.2. The general case $d \geqslant 1$. Let us now investigate the general case, where $d \geqslant 1$. Let $N_0 \in \mathbb{N}$ such that

$$N_0 \geqslant \max\{2^{q/(d(q-p))} - 1, 2^{s/(d(s-r))} - 1\}.$$

Let I^β_α be defined as before and define $(I^\beta_\alpha)^d$ as the Cartesian product

$$(I^\beta_\alpha)^d = I^\beta_a \times \ldots \times I^\beta_a,$$

where $\alpha, \beta \in \mathbb{R}$. Define

$$f(u) = \chi_{(I^1_1)^d}(u) + \chi_{I^{n+1}_1\times(I^1_1)^{d-1}}(u), \quad u \in \mathbb{R}^d,$$

and

$$g(u)=\chi_{(I_1^1)^d}(u)-\chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(u),\quad u\in\mathbb{R}^d.$$

Next, we compute the norm of f by considering three cases of cubes whose intersections with the support of f are non-empty. For the first case, that is, the cube Q with $\ell(Q)\leqslant 1$ such that $Q\cap(I_1^1)^d\neq\emptyset$, the maximum value of $\mathfrak{m}_{p,q}(f;Q)$ is 1 and it is attained when $Q=(I_1^1)^d$. For the second case, that is, the cube Q with $\ell(Q)\leqslant 1$ such that $Q\cap I_1^{n+1}\times(I_1^1)^{d-1}\neq\emptyset$, the maximum value of $\mathfrak{m}_{p,q}(f;Q)$ is 1 and it is attained when $Q=I_1^{n+1}\times(I_1^1)^{d-1}$.

For the last case, that is, the cube Q such that $Q\cap(I_1^1\cup I_1^{n+1})\times(I_1^1)^{d-1}\neq\emptyset$, the largest value of $\mathfrak{m}_{p,q}(f;Q)$ is attained when $Q=I_{n+1}^{n+1}\times\prod\limits_{i=1}^{d-1}I_{n+1}^{n+1+\alpha_i}$, $-n\leqslant\alpha_i\leqslant 0$. For such Q,

$$\mathfrak{m}_{p,q}(f;Q)=|Q|^{1/q-1/p}\Big(\int_Q|f(u)|^p\,\mathrm{d}u\Big)^{1/p}=((n+1)^d)^{1/q-1/p}2^{1/p}.$$

Therefore,

$$\|f\|_{\mathcal{M}_q^p}=\max\{1,((n+1)^d)^{1/q-1/p}2^{1/p}\}.$$

Thus, by choosing $n\geqslant N_0$, we have $\|f\|_{\mathcal{M}_q^p}=1$. By the same reasoning, we get $\|f\|_{\mathcal{M}_s^r}=\|g\|_{\mathcal{M}_q^p}=\|g\|_{\mathcal{M}_s^r}=1$.

Let us now define

$$\overline{f}(u,v)=f(u)f(v)\quad\text{and}\quad\overline{g}(u,v)=g(u)g(v),\quad u,v\in\mathbb{R}^d, \tag{2.6}$$

so that

$$\|\overline{f}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}=\|f\|_{\mathcal{M}_q^p}\|f\|_{\mathcal{M}_s^r}=1 \tag{2.7}$$

and

$$\|\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}=\|g\|_{\mathcal{M}_q^p}\|g\|_{\mathcal{M}_s^r}=1. \tag{2.8}$$

As before, in order to calculate $\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}$, one may observe that for $u\in\mathbb{R}^d$ we have

$$\begin{aligned}\|\overline{f}+\overline{g}\|_{\mathcal{M}_s^r,u}&=\begin{cases}2, & v\in(I_1^1)^d,\\ 2, & v\in I_1^{n+1}\times(I_1^1)^{d-1},\\ 0, & v\in\mathbb{R}^d\setminus(I_1^1\cup I_1^{n+1})\times(I_1^1)^{d-1},\end{cases}\\ &=2\chi_{(I_1^1)^d}(v)+2\chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(v)=2f(v).\end{aligned}$$

Hence, we get

$$\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}=\|\|(\overline{f}+\overline{g})(u,v)\|_{\mathcal{M}_s^r,u}\|_{\mathcal{M}_q^p,v}=\|2f\|_{\mathcal{M}_q^p}=2. \tag{2.9}$$

Meanwhile, one may also observe that for $u \in \mathbb{R}^d$ we have

$$
\begin{aligned}
\|\overline{f}-\overline{g}\|_{\mathcal{M}_s^r,u} &= \begin{cases} 2, & v \in (I_1^1)^d, \\ 2, & v \in I_1^{n+1} \times (I_1^1)^{d-1}, \\ 0, & v \in \mathbb{R}^d \setminus (I_1^1 \cup I_1^{n+1}) \times (I_1^1)^{d-1}, \end{cases} \\
&= 2\chi_{(I_1^1)^d}(v) + 2\chi_{I_1^{n+1}\times (I_1^1)^{d-1}}(v) = 2f(v).
\end{aligned}
$$

As a result, we obtain

$$
\|\overline{f}-\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|\|(\overline{f}-\overline{g})(u,v)\|_{\mathcal{M}_s^r,u}\|_{\mathcal{M}_q^p,v} = \|2f\|_{\mathcal{M}_q^p} = 2. \tag{2.10}
$$

From equations (2.7)–(2.10) we deduce that

$$
C_{NJ}(\mathcal{M}_q^p(\mathcal{M}_s^r)) \geqslant \frac{\|\overline{f}+\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{f}-\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)}}{2(\|\overline{f}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)} + \|\overline{g}\|^2_{\mathcal{M}_q^p(\mathcal{M}_s^r)})} = \frac{2^2+2^2}{2(1+1)} = 2,
$$

as desired.

For the James constant, consider the functions $\overline{f}$ and $\overline{g}$ defined by (2.6). From equations (2.9) and (2.10) it follows that

$$
\begin{aligned}
&C_J(\mathcal{M}_q^p(\mathcal{M}_s^r)) \\
&\quad = \sup\{\min\{\|\overline{f}+\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}, \|\overline{f}-\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)}\} \colon \|\overline{f}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|\overline{g}\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 1\} \\
&\quad = 2.
\end{aligned}
$$

To compute the Dunkl-Williams constant, let $\overline{f}$ and $\overline{g}$ be as in (2.6). For $\delta > 0$, define

$$
w = (1+\delta)h + (1-\delta)k,
$$

where $h = \overline{f}+\overline{g}$ and $k = \overline{f}-\overline{g}$. Here $\|h\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = \|k\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 2$.

As in the case, where $d = 2$, we claim that $\|w\|_{\mathcal{M}_q^p(\mathcal{M}_s^r)} = 2+2\delta$. First, one may observe that

$$
w = (1+\delta)h + (1-\delta)k = 2\overline{f} + 2\delta\overline{g},
$$

whence

$$
w(u,v) = 2f(u)f(v) + 2\delta g(u)g(v).
$$

Now let $u \in \mathbb{R}^d$. For $v \in (I_1^1)^d$, we have

$$
\begin{aligned}
w(u,v) &= 2f(u) + 2\delta g(u) \\
&= 2(\chi_{(I_1^1)^d}(u) + \chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(u)) + 2\delta(\chi_{(I_1^1)^d}(u) - \chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(u)) \\
&= \begin{cases} 2+2\delta, & u \in (I_1^1)^d, \\ 2-2\delta, & u \in I_1^{n+1} \times (I_1^1)^{d-1}, \\ 0, & u \in \mathbb{R}^d \setminus (I_1^1 \cup I_1^{n+1}) \times (I_1^1)^{d-1}. \end{cases}
\end{aligned}
$$

This implies that

$$\|w(u,v)\|_{\mathcal{M}^r_s,u} = 2+2\delta.$$

For $v \in \mathbb{R}^d \setminus (I_1^1 \cup I_1^{n+1}) \times (I_1^1)^{d-1}$, we have $w(u,v)=0$, so that

$$\|w(u,v)\|_{\mathcal{M}^r_s,u} = 0.$$

For $v \in I_1^{n+1} \times (I_1^1)^{d-1}$, we have

$$\begin{aligned} w(u,v) &= 2f(u) - 2\delta g(u) \\ &= 2(\chi_{(I_1^1)^d}(u) + \chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(u)) - 2\delta(\chi_{(I_1^1)^d}(u) - \chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(u)) \\ &= \begin{cases} 2-2\delta, & u \in (I_1^1)^d, \\ 2+2\delta, & u \in I_1^{n+1} \times (I_1^1)^{d-1}, \\ 0, & u \in \mathbb{R}^d \setminus (I_1^1 \cup I_1^{n+1}) \times (I_1^1)^{d-1}. \end{cases} \end{aligned}$$

It follows that

$$\|w(u,v)\|_{\mathcal{M}^r_s,u} = 2+2\delta.$$

We summarize that

$$\begin{aligned} \|w(u,v)\|_{\mathcal{M}^r_s,u} = &\begin{cases} 2+2\delta, & v \in (I_1^1)^d, \\ 2+2\delta, & v \in I_1^{n+1} \times (I_1^1)^{d-1}, \\ 0, & v \in \mathbb{R}^d \setminus (I_1^1 \cup I_1^{n+1}) \times (I_1^1)^{d-1}), \end{cases} \\ &= (2+2\delta)\chi_{(I_1^1)^d}(v) + (2+2\delta)\chi_{I_1^{n+1}\times(I_1^1)^{d-1}}(v) = 2+2\delta)f(v). \end{aligned}$$

As a consequence, we get

$$\|w\|_{\mathcal{M}^p_q(\mathcal{M}^r_s)} = \|\|w(u,v)\|_{\mathcal{M}^r_s,u}\|_{\mathcal{M}^p_q,v} = (2+2\delta)\|f\|_{\mathcal{M}^p_q} = 2+2\delta,$$

as claimed.

Therefore, as in the case where $d=2$, we obtain

$$\frac{\|h+k\|_X + \|w\|_X}{\delta\|k-h\|_X}\left\|\frac{h+k}{\|h+k\|_X} - \frac{w}{\|w\|_X}\right\|_X = \frac{4+2\delta}{1+\delta},$$

where $X = \mathcal{M}^p_q(\mathcal{M}^r_s)$. Taking $\delta \to 0^+$, we get $C_{DW}(\mathcal{M}^p_q(\mathcal{M}^r_s)) = 4$ as expected.

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