3 Day School on Harmonic and Functional Analysis

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In this 3-Day School on Harmonic and Functional Analysis, we shall discuss:

1. L^p spaces
2. Operators on L^p spaces
3. Morrey spaces
4. Operators on Morrey spaces
5. More about Morrey spaces
6. Boundedness of certain operators on (weak) Morrey spaces

Key words: Lebesgue spaces, Morrey spaces, bounded operators.

LECTURE 6: MORE ABOUT MORREY SPACES

Here we shall discuss:

- Further Geometric Properties of Morrey Spaces

Many references will be given, for further reading.

Morrey Spaces

Definition, Version 2

For $1 \leq p \leq q < \infty$, the Morrey space $M^p_q = M^p_q(\mathbb{R}^d)$ is defined to be the space of all $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ for which

$$\|f\|_{M^p_q} := \sup_{B=B(a,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(y)|^p dy \right)^{\frac{1}{p}} \tag{1}$$

is finite, where $|B|$ denotes the (Lebesgue) measure of B in $\mathbb{R}^d$.

Note that $M^p_p = L^p$.

Uniformly Convexity and Strcitly Convexity

A Banach space $(X, \|\cdot\|)$ is said to be *strictly convex* (SC) iff for every $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\|x + y\| < 2$.

A Banach space $(X, \|\cdot\|)$ is said to be *uniformly convex* (UC) iff for every $0 < \epsilon \leq 2$ there exists $\delta > 0$ such that for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, we have $\|x + y\| \leq 2(1 - \delta)$.

Note that UC $\Rightarrow$ SC, but SC $\not\Rightarrow$ UC.

In approximation theory, SC guarantees the uniqueness of best approximation but not the existence; while UC guarantees the existence and uniqueness of best approximation.

Thm 1. For $1 \leq p < q < \infty$, M^p_q are not uniformly convex

As we shall see later, one can find $f, g \in M^p_q$ such that

$$\|f\|_{M^p_q} = \|g\|_{M^p_q} = 1$$

and

$$\|f + g\|_{M^p_q} = 2.$$

Hence M^p_q is not strictly convex, and accordingly M^p_q is not uniformly convex.

Note: For $1 \leq p < q < \infty$, M^p_q is geometrically similar to L^1 or L^∞.

Uniformly Non-Squareness

A Banach space $(X, \|\cdot\|)$ is said to be *non-square* (NS) iff for every $x, y \in X$ with $\|x\| = \|y\| = 1$, we have either $\|x + y\| < 2$ or $\|x - y\| < 2$ [James, 1964].

To be non-square, there cannot be $x, y \in X$ with $\|x\| = \|y\| = 1$ such that $\|x + y\| = \|x - y\| = 2$ (or, equivalently, there cannot be $u, v \in X$ with $\|u\| = \|v\| = 1$ such that $\|u + v\| = \|u - v\| = 1$).

A Banach space $(X, \|\cdot\|)$ is said to be *uniformly non-square* (UNS) iff there exists $\delta > 0$ such that for every $x, y \in X$ with $\|x\| = \|y\| = 1$, we have either $\|x + y\| \leq 2(1 - \delta)$ or $\|x - y\| \leq 2(1 - \delta)$.

Note that UC $\Rightarrow$ UNS. Moreover, X is reflexive when X is UNS [James, 1964].

Ref: R.C. James, Ann. Math. 80 (1964), 542–550.

Thm 2. For $1 \leq p < q < \infty$, M^p_q are not uniformly non-square

One can construct $F, G \in M^p_q$ with $\|F\| = \|G\| = 1$ such that $\|F+G\|_{M^p_q} = \|F-G\|_{M^p_q} = 2$, and conclude that M^p_q are not uniformly non-square.

To be precise, let $f(x) \equiv |x|^{-\frac{d}{q}}$, $h(x) \equiv \chi_{(0,1)}(|x|)f(x)$, $k(x) \equiv f(x) - h(x)$, and $g(x) = -f(x) + 2h(x)$. Then, $\|f\|_{M^p_q} = \|g\|_{M^p_q} = \|h\|_{M^p_q} = \|k\|_{M^p_q} = c_q$, say. Accordingly,

$$\min\{\|f+g\|_{M^p_q}, \|f-g\|_{M^p_q}\} = \min\{\|2h\|_{M^p_q}, \|2k\|_{M^p_q}\} = 2c_q.$$

Thus, $F = c_q^{-1} f$ and $G = c_q^{-1} g$ are the functions we look for.

Ref: H. Gunawan, E. Kikianty, Y. Sawano, and C. Schwanke, Bull. Kor. Math. Soc. 56 (2019), 1569–1575.

Uniformly Non-ℓ_n^1-ness

A Banach space $(X, \|\cdot\|)$ is said to be *uniformly non-ℓ_n^1* iff there exists a $\delta > 0$ such that for every $x_1, \ldots, x_n \in X$ with $\|x_1\| = \cdots = \|x_n\| = 1$, we have

$$\min \|x_1 \pm \cdots \pm x_n\| \leq n(1-\delta),$$

where the minimum is taken over all combinations of $\pm$ signs [Beauzamy, 1985].

If, for every $\delta > 0$, we can find $x_1, \ldots, x_n \in X$ with $\|x_1\| = \cdots = \|x_n\| = 1$ such that $\|x_1 \pm \cdots \pm x_n\| > n(1-\delta)$ for all choices of $\pm$ signs, then X is not uniformly non-ℓ_n^1.

Note that if X is uniformly non-ℓ_n^1, then X is uniformly non-ℓ_{n+1}^1.

The uniformly non-ℓ_n^1-ness property is weaker than the so-called *uniformly n-convex property*.

Thm 3. For $1 \le p < q < \infty$, M^p_q are not uniformly non-ℓ^1_n

In [Gunawan et al., 2021], it is shown that, for $1 \le p < q < \infty$, M^p_q are not uniformly non-ℓ^1_n. This result can be reproved by using analogous results in discrete Morrey spaces $\ell^p_q = \ell^p_q(\mathbb{Z}^d)$ and the fact that ℓ^p_q can be embedded into M^p_q via the mapping

$$\tilde{x}(t) := \sum_k x_k \chi_{Q[k-\mathbf{1},k)}(t), \quad x = (x_k)_{k\in\mathbb{Z}^d} \in \ell^p_q,$$

where $Q[k-\mathbf{1},k)$ is the cube $[k_1-1,k_1)\times\cdots\times[k_d-1,k_d)$ in $\mathbb{R}^d$, for each $k=(k_1,\ldots,k_d)\in\mathbb{Z}^d$.

Note: The result on ℓ^p_q can be found in [Adam and Gunawan, 2022]. The space ℓ^p_q will be discussed in Lecture 6.

Discrete Morrey Spaces
Definition

For $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $N \in \mathbb{N} \cup \{0\}$, write

$$S_{m,N} := \{k \in \mathbb{Z}^d \; : \; \|k - m\|_\infty \le N\},$$

where $\|m\|_\infty = \max\{|m_1|, \dots, |m_d|\}$.

Note that the cardinality of $S_{m,N}$ is then $|S_{m,N}| = (2N+1)^d$.

For $1 \le p \le q < \infty$, we define the *discrete Morrey space* $\ell_q^p = \ell_q^p(\mathbb{Z}^d)$ to be the set of all sequences $x : \mathbb{Z}^d \to \mathbb{R}$ such that

$$\|x\|_{\ell_q^p} := \sup_{m,N} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \Big(\sum_{k \in S_{m,N}} |\xi_k|^p \Big)^{\frac{1}{p}},$$

where $x = (\xi_k)_{k \in \mathbb{Z}^d}$ [Gunawan et al., 2018].

Thm 4. For $1 \le p < q < \infty$, ℓ^p_q are not uniformly non-square

Choose $J \ge \frac{q}{d(q-p)}$, and define $x = (\xi_k)_{k\in\mathbb{Z}^d}$ and $y = (\nu_k)_{k\in\mathbb{Z}^d}$ where

$$\xi_k := \begin{cases} 1, & \text{if } k = (1,1,\dots,1),\ (2^J,1,\dots,1), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_k := \begin{cases} 1, & \text{if } k = (1,1,\dots,1), \\ -1, & \text{if } k = (2^J,1,\dots,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|x\|_{\ell^p_q} = \|y\|_{\ell^p_q} = \max\left\{1, 2^{Jd(\frac{1}{q}-\frac{1}{p})} 2^{\frac{1}{p}}\right\} = 1$ (because of the choice of J).

Meanwhile $\|x+y\|_{\ell^p_q} = \|x-y\|_{\ell^p_q} = 2$.

Thm 5. For $1 \le p < q < \infty$, ℓ_q^p are not uniformly non-ℓ_n^1

Choose $J \ge \frac{q}{d(q-p)}$, and define $x = (\xi_k)_{k\in\mathbb{Z}^d}$ where

$$\xi_k := \begin{cases} 1, & \text{if } k = (1,1,\ldots,1),\ (2^J,1,\ldots,1),\ldots,(2^{(2^{n-1}-1)J},1,\ldots,1), \\ 0, & \text{otherwise.} \end{cases}$$

Note that the distances between two consecutive values of k's for which $x_k = 1$ are getting larger as the points get further away from 0. Hence, as we look for the supremum, we need only to consider

$$v_j = |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \Big(\sum_{k\in S_{m,N}} |\xi_k|^p \Big)^{\frac{1}{p}}, \quad j = 0,1,\ldots,2^{n-1}-1,$$

where $S_{m,N}$ is the smallest set that contains $(1,1,\ldots,1)$ and $(2^{jJ},1,\ldots,1)$.
By the choice of J, $\|x\|_{\ell_q^p} = 1$.
To prove our claim, we need to construct the other $n-1$ sequences.

T5. For $1 \le p < q < \infty$, ℓ^p_q are not uniformly non-octahedral

To illustrate the proof, let us consider the case where $n = 3$, where the result states that ℓ^p_q are not *uniformly non-octahedral*.

Here we need to construct $x = (\xi_k)_{k\in\mathbb{Z}^d}, y = (\nu_k)_{k\in\mathbb{Z}^d}$ and $z = (\zeta_k)_{k\in\mathbb{Z}^d}$ such that

$$\|x \pm y \pm z\|_{\ell^p_q} = 3$$

for all choices of $\pm$ signs.

Note: $x \pm y \pm z$ are the four diagonals of the parallepiped spanned by x, y and z, Taking account $-x \pm y \pm z$ together, we have eight vectors, which corresponds to eight faces of an octahedral.

For $n = 2$, $x \pm y$ are the two diagonals of the parallelogram spanned by x and y.

We define

$$\xi_k := \begin{cases} 1, & \text{if } k = (1,1,\ldots,1),\ (2^J,1,\ldots,1),(2^{2J},1,\ldots,1),(2^{3J},1,\ldots,1), \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_k := \begin{cases} 1, & \text{if } k = (1,1,\ldots,1),\ (2^J,1,\ldots,1), \\ -1, & \textit{if } k = (2^{2J},1,\ldots,1),\ (2^{3J},1,\ldots,1), \\ 0, & \text{otherwise;} \end{cases}$$

$$\zeta_k := \begin{cases} 1, & \text{if } k = (1,1,\ldots,1),\ (2^{2J},1,\ldots,1), \\ -1, & \textit{if } k = (2^J,1,\ldots,1),\ (2^{3J},1,\ldots,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, one may observe that $\|x \pm y \pm z\|_{\ell^p_q} = 3$.

Von Neumann-Jordan Constant and James Constant

For a Banach space $(X, \|\cdot\|)$, the Von Neumann-Jordan constant is defined by

$$C_{NJ}(X) := \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \,:\, x, y \neq 0\right\},$$

while the James constant is defined by

$$C_J(X) := \sup\{\min\{\|x+y\|, \|x-y\|\} \,:\, \|x\| = \|y\| = 1\}.$$

It is known that, for any Banach space X, we have $1 \leq C_{NJ}(X) \leq 2$ and $\sqrt{2} \leq C_J(X) \leq 2$.

Now let $X = \ell_q^p$ or M_q^p with $1 \leq p < q < \infty$. As we can find $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$ such that $\|x+y\|_X = \|x-y\|_X = 2$, we conclude that $C_{NJ}(X) = C_J(X) = 2$ [Gunawan et al., 2019].

n-th Von Neumann-Jordan Constant and n-th James Constant

For a Banach space $(X, \|\cdot\|)$, the n-th Von Neumann-Jordan constant is defined by

$$C_{NJ}^{(n)}(X) := \sup\left\{ \frac{\sum_{\pm} \|x_1 \pm \cdots \pm x_n\|^2}{2^{n-1}\sum_{i=1}^{n} \|x_i\|^2} \,:\, x_i \neq 0,\ i = 1, \dots, n \right\},$$

while the n-th James constant is defined by

$$C_{J}^{(n)}(X) := \sup\{\min \|x_1 \pm \cdots \pm x_n\| \,:\, \|x_i\| = 1,\ i = 1, \dots, n\}.$$

It is known that for any Banach space X, we have $1 \leq C_{NJ}^{(n)}(X) \leq n$ [Kato et al., 1988] and $\sqrt{n} \leq C_{J}^{(n)}(X) \leq n$ [Maligandra et al., 2007].

Theorem 6. For $X = \ell_q^p$ or M_q^p with $1 \leq p < q < \infty$, we have $C_{NJ}^{(n)}(X) = C_{J}^{(n)}(X) = n$.

Concluding Remarks

In this lecture, we have proved that, for $1 \leq p < q < \infty$, ℓ_q^p are not uniformly non-ℓ_n^1 spaces, and accordingly M_q^p are not uniformly non-ℓ_n^1 spaces.

Consequently, these spaces are not uniformly n-convex for $n \geq 2$.

Related to the above results, we have found that $C_{NJ}^{(n)}(X) = C_J^{(n)}(X) = n$ for both $X = \ell_q^p$ and $X = M_q^p$, with $1 \leq p < q < \infty$.

The values of the constants for both M_q^p and ℓ_q^p were obtained previously in [Gunawan et al., 2021] and [Adam and Gunawan, 2022], respectively.

What is new is the set of functions and sequences, respectively, that we construct to confirm the values of the constants.

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