BOUNDEDNESS OF THE HARDY-LITTLEWOOD MAXIMAL OPERATOR, FRACTIONAL INTEGRAL OPERATORS, AND CALDERÓN-ZYGMUND OPERATORS ON GENERALIZED WEIGHTED MORREY SPACES

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Abstract. In this paper we investigate the boundedness of three classical operators, namely the Hardy-Littlewood maximal operator, fractional integral operators, and Calderón-Zygmund operators, on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. We prove that each of the three operators is bounded on these spaces under some assumptions.

1. Introduction

We shall discuss the boundedness of three classical operators, namely the Hardy-Littlewood maximal operator, fractional integral operators, and Calderón-Zygmund operators, on generalized weighted Morrey spaces. Throughout this paper, we denote by $B(a, r)$ an open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$. For a set E in $\mathbb{R}^n$, we denote by E^c the complement of E. Moreover, if E is a measurable set in $\mathbb{R}^n$, then $|E|$ denotes the Lebesgue measure of E. The Hardy-Littlewood maximal operator M and fractional maximal operator M_α, where $0 \le \alpha < n$, are defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n,$$

and

$$M_\alpha f(x) := \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n,$$

for locally integrable functions f on $\mathbb{R}^n$. It is well-known that M is bounded on Lebesgue spaces $L^p = L^p(\mathbb{R}^n)$ for $1 < p \le \infty$, and from L^1 to the weak Lebesgue space $WL^1 = WL^1(\mathbb{R}^n)$, see e.g. [8, 20, 22].

For $0 < \alpha < n$, we also know the Riesz potential or the fractional integral operator I_α, which is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

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for suitable functions f on $\mathbb{R}^n$. The operator I_α is bounded from L^p to L^q for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, see e.g. [22]. Since from the definitions we have

$$M_\alpha f(x) \le C_n I_\alpha(|f|)(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where C_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$, it thus follows that the operator M_α is also bounded from L^p to L^q for $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The next operator that we discuss is the Calderón-Zygmund operator. Let $T = T_K$ be a linear operator from the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'$ which is L^2-bounded and, for each $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y) f(y) dy, \quad x \notin \mathrm{supp}(f),$$

where $K = K(\cdot,\cdot)$ is the standard kernel defined on $\mathbb{R}^n \times \mathbb{R}^n$ except for the diagonal $\{(x,x) : x \in \mathbb{R}^n\}$ with the following properties: there exists a constant $A > 0$ for which

$$|K(x,y)| \le \frac{A}{|x-y|^n}, \quad x \ne y,$$

and, for some $\delta > 0$,

$$|K(x,y) - K(x',y)| \le \frac{A|x-x'|^\delta}{(|x-y| + |x'-y|)^{n+\delta}}, \quad |x - x'| \le \frac{1}{2} \max(|x-y|, |x'-y|)$$

and

$$|K(x,y) - K(x,y')| \le \frac{A|y-y'|^\delta}{(|x-y| + |x-y'|)^{n+\delta}}, \quad |y - y'| \le \frac{1}{2} \max(|x-y|, |x-y'|).$$

The operator T is called the Calderón-Zygmund operator, which was introduced by Coifman and Meyer in 1979 [3]. The operator is bounded on L^p for $1 < p < \infty$ and from L^1 to WL^1 [8].

Let us now discuss about Morrey spaces that we shall work on. For $1 \le p < \infty$ and $0 \le \lambda < n$, the classical Morrey space $\mathcal{M}^{p,\lambda} = \mathcal{M}^{p,\lambda}(\mathbb{R}^n)$, equipped with the following norm

$$\|f\|_{\mathcal{M}^{p,\lambda}} := \sup_{a \in \mathbb{R}^n, r>0} \frac{1}{r^\lambda} \left(\int_{B(a,r)} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

was first introduced in [14]. The same space may be denoted by $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^n)$, equipped with

$$\begin{aligned} \|f\|_{\mathcal{M}_p^q} &:= \sup_{a \in \mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{B(a,r)} |f(x)|^p dx \right)^{\frac{1}{p}} \\ &= \sup_{a \in \mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L^p(B(a,r))} \end{aligned}$$

where $1 \le p \le q < \infty$, as used widely in, for examples, [10, 12, 20]. Note that if we set $p = q$, then $\mathcal{M}_q^p = L^p$. In companion with $\mathcal{M}_q^p$, one may also define the weak Morrey space $W\mathcal{M}_q^p = W\mathcal{M}_q^p(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{W\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{|B(a,r)|^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{WL^p(B(a,r))} < \infty,$$

where $\|f\|_{WL^p(B(a,r)} := \sup\limits_{\gamma>0} \gamma\,|\{x \in B(a,r) : |f(x)| > \gamma\}|^{1/p}$. The last two definitions were used in, for example, [11].

According to [6], T is bounded on $\mathcal{M}_p^q$ for $1 < p \leq q < \infty$ and is bounded from $\mathcal{M}_1^q$ to $W\mathcal{M}_1^q$ for $1 \leq q < \infty$. In addition, M is bounded on $\mathcal{M}_p^q$ for $1 < p \leq q < \infty$ and is bounded from $\mathcal{M}_1^q$ to $W\mathcal{M}_1^q$ for $1 \leq q < \infty$ [2]. Moreover, I_α is bounded from one Morrey space to another under certain conditions [1, 18].

In [13, 17], the Morrey space $\mathcal{M}_q^p$ was generalized to $\mathcal{M}_\psi^p = \mathcal{M}_\psi^p(\mathbb{R}^n)$, which consists of all locally integrable functions f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{\mathcal{M}_\psi^p} := \sup_{a\in\mathbb{R}^n} \left(\frac{1}{\psi(B(a,r))}\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty.$$

Here ψ is a function from $(0, \mathbb{R}^n \times \infty)$ to $(0,\infty)$ satisfying certain conditions. Moreover, the weak generalized Morrey space $W\mathcal{M}_\psi^p = W\mathcal{M}_\psi^p(\mathbb{R}^n)$ where $1 < p < \infty$ was defined as the set of all functions f for which there exists a constant $C > 0$ such that

$$\frac{\gamma^p}{\psi(B)}|\{x \in B : |f(x)| > \gamma\}| \leq C$$

for every ball $B = B(a,r)$ and $\gamma > 0$. We can see that if we set $\psi(B(a,r)) = |B(a,r)|^{1-\frac{p}{q}}$ where $1 \leq q < \infty$, then $\mathcal{M}_\psi^p = \mathcal{M}_q^p$. In [17], Nakai investigated the sufficient conditions on the function ψ to ensure the boundedness of the operator M, T, and I_α on these spaces. Similar results are obtained by Mizuhara [13], where ψ was assumed to be a growth function satisfying doubling condition with a doubling constant $1 \leq D = D(\psi) < 2^n$.

In [9], Guliyev defined generalized Morrey spaces $\mathcal{M}_\phi^p = \mathcal{M}_\phi^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{M}_\phi^p} := \sup_{a\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\phi(a,r)} \|f\|_{L^p(B(a,r))}$$

and also defined generalized weak Morrey spaces $W\mathcal{M}_\phi^p = W\mathcal{M}_\phi^p(\mathbb{R}^n)$ with the norm

$$\|f\|_{W\mathcal{M}_\phi^p} = \sup_{a\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\phi(a,r)} \|f\|_{WL^p(B(a,r))}.$$

In contrast with Nakai's approach, Guliyev did not use the doubling condition to prove the boundedness of the operators M and T on these spaces. Furthermore, Guliyev also investigated the boundedness of M and T from $\mathcal{M}_{\phi_1}^p$ to $\mathcal{M}_{\phi_2}^p$ for $1 < p < \infty$, and from $\mathcal{M}_{\phi_1}^1$ to $W\mathcal{M}_{\phi_2}^1$ for $p = 1$, for some functions ϕ_1 and ϕ_2 on $\mathbb{R}^n \times (0,\infty)$. To be precise, he obtained the following theorem.

Theorem 1.1. [9] *Let $1 \leq p < \infty$ and the functions $\phi_1(a,r)$ and $\phi_2(a,r)$ satisfy*

$$\int_r^\infty \phi_1(a,t)\frac{dt}{t} \leq C\phi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$ where C does not depend on a and r. Then, M and T are bounded from $\mathcal{M}^p_{\phi_1}$ to $\mathcal{M}^p_{\phi_2}$ for $1 < p < \infty$ and are bounded from $\mathcal{M}^1_{\phi_1}$ to $W\mathcal{M}^1_{\phi_2}$.

Similar to Guliyev's definitions, we may also defined the generalized Morrey space $\mathcal{M}^p_\phi$ as the set of all locally integrable functions f on $\mathbb{R}^n$ such that $\|f\|_{\mathcal{M}^p_\phi}$ is finite where

$$\begin{aligned}\|f\|_{\mathcal{M}^p_\phi} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi(a,r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx \right)^{\frac{1}{p}} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi(a,r)} \cdot \frac{1}{|B(a,r)|^{\frac{1}{p}}} \|f\|_{L^p(B(a,r))}\end{aligned}$$

for $1 \le p < \infty$ and a positive function ϕ on $\mathbb{R}^n \times (0,\infty)$. Moreover, we may also defined generalized weak Morrey space $\mathcal{M}^p_\phi$ as the set of all locally integrable functions f on $\mathbb{R}^n$ such that $\|f\|_{W\mathcal{M}^p_\phi} < \infty$ where

$$\|f\|_{\mathcal{M}^p_\phi} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi(a,r)} \cdot \frac{1}{|B(a,r)|^{\frac{1}{p}}} \|f\|_{WL^p(B(a,r))}.$$

The purpose of this article is to investigate the boundedness of M, I_α, and T on generalized weighted Morrey spaces and weighted generalized weak Morrey spaces. The spaces were defined by simply replacing $|B(a,r)|$ with $w(B(a,r)) := \int_{B(a,r)} w(x)dx$ for some weight function w which we shall discuss in the next section. Our results, which are presented in Section 3-5, generalize the previous results obtained by Guliyev.

2. A_p Weights and Generalized Weighted Morrey Spaces

In this section, we discuss A_p weights, weighted Lebesgue spaces, and weighted Morrey spaces. We also present the definition of generalized weighted Morrey spaces, generalized weighted weak Morrey spaces, and some lemmas that we shall use to prove the main results about the boundedness of the three classical operators on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

A weight w is a non-negative locally integrable function on $\mathbb{R}^n$ taking values in the interval $(0,\infty)$ almost everywhere. The weight class that we use in this article is the Muckenhoupt class A_p (see, e.g, [7]).

Definition 2.1. For $1 < p < \infty$, we denote by A_p the set of all weights w on $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)dx \right) \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \le C$$

for every ball $B(a,r)$ in $\mathbb{R}^n$. For $p = 1$, we denote by A_1 the set of all weights w for which there exists a constant $C > 0$ such that

$$\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)dx \le C\|w\|_{L^\infty(B(a,r))}$$

for every ball $B(a,r)$ in $\mathbb{R}^n$.

Remark 2.2. The last inequality is equivalent to the following

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)dx\right)\cdot \|w^{-1}\|_{L^\infty(B(a,r))} \le C$$

for every ball $B(a,r)$ in $\mathbb{R}^n$.

Theorem 2.3. [7] *For each $1 \le p < \infty$ and $w \in A_p$, there exists $C > 0$ such that*

$$\frac{w(B)}{w(E)} \le C\left(\frac{|B|}{|E|}\right)^p$$

for every ball B and measurable sets $E \subseteq B$ where

$$w(B) = \int_B w(x)dx.$$

Associated to a weight function $w \in A_p$ with $1 \le p < \infty$, we define the weighted Lebesgue space $L^{p,w} = L^{p,w}(\mathbb{R}^n)$ to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{L^{p,w}} := \int_{\mathbb{R}^n} |f(x)|^p w(x)dx < \infty.$$

In addition, we denote by $WL^{p,w} = WL^{p,w}(\mathbb{R}^n)$ the weighted weak Lebesgue space that consists of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{WL^{p,w}} := \sup_{\gamma>0} \gamma w(\{x \in \mathbb{R}^n : |f(x)| > \gamma\})^{\frac{1}{p}} < \infty.$$

Notice that if w is a constant function a.e., then we have that $L^{p,w} = L^p$ and $WL^{p,w} = WL^p$. We note from [7] that $w \in A_p$ if and only if M is bounded on $L^{p,w}$ for $1 < p < \infty$ and $w \in A_1$ if and only if M is bounded from $L^{1,w}$ to $WL^{1,w}$.

Related to the fractional integral operator I_α, we have another class of weights $A_{p,q}$.

Definition 2.4. [16, 20] Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. We denote by $A_{p,q}$ the collection of all weight functions w satisfying

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{\frac{1}{q}}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^{-p'} dx\right)^{\frac{1}{p'}} \le C$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$ where C is a constant independent of a and r. For $p = 1$ and $q > 1$, we denote by $A_{1,q}$ the collection of weight functions w for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{1/q} \le C\|w\|_{L^\infty(B(a,r))}$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$.

One may observe that $w \in A_{p,q}$ if and only if $w^q \in A_{q/p'+1}$ for $1 \le p < q < \infty$ (see [5]). Moreover, we have the following proposition.

Proposition 2.5. *Let $1 \le p < q < \infty$. If $w \in A_{p,q}$, then $w^p \in A_p$ and $w^q \in A_q$.*

Proof. First suppose that $w \in A_{p,q}$, where $1 < p < q < \infty$. Then,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^q dy\right)^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^{-p'} dy\right)^{-\frac{1}{p'}} \le C$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$. Since, by Hölder's inequality,

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^p dy\right)^{\frac{1}{p}} \le \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^q dy\right)^{\frac{1}{q}},$$

and $p' = \frac{p}{p-1}$, we obtain

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^p dy\right) \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^{-\frac{p}{p-1}} dy\right)^{\frac{1}{p-1}} \le C$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, which tells us that $w^p \in A_p$.

Next, if $w \in A_{1,q}$ for $1 < q < \infty$, then

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x)^q dx\right)^{\frac{1}{q}} \le C\|w\|_{L^\infty(B(a,r))}$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$. By Hölder's inequality, we have

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} w(x) dx \le C\|w\|_{L^\infty(B(a,r))}.$$

This means that $w \in A_1$.

Finally, to show that $w^q \in A_q$ for $1 \le p < q$, we observe that $q' < p'$, and hence

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^q dy\right)^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^{-q'} dy\right)^{\frac{1}{q'}}$$

$$\le \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^q dy\right)^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^{-p'} dy\right)^{\frac{1}{p'}} \le C.$$

(One should interpret the above inequality appropriately for the case where $p = 1$.) Taking the q-th power, we obtain

$$\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} w(y)^q dy\right) \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} (w(y)^q)^{-\frac{1}{q-1}} dy\right)^{q-1} \le C,$$

whence $w^q \in A_q$, and so the proof is complete. □

We rewrite the following results of Muckenhoupt [15] for M, Muckenhoupt and Wheeden [16] for I_α, Coifman and Fefferman [4], Garcia-Cuerva and Rubio de Francia [7], and E. Sawyer [21] for T on weighted Lebesgue spaces:

Theorem 2.6. [7] *Let $1 < p < \infty$. Then, M is bounded on $L^{p,w}$ if $w \in A_p$. Moreover, M is bounded from $L^{1,w}$ to $WL^{1,w}$ if $w \in A_1$.*

Theorem 2.7. [16] *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Then, the operator I_α is bounded from L^{p,w^p} to L^{q,w^q}. Moreover, if $w \in A_{1,p}$ with $1/q = 1 - \alpha/n$, then I_α is bounded from $L^{1,w}$ to WL^{q,w^q}.*

Theorem 2.8. [4, 7, 21] *Let $1 \le p < \infty$. Then, T is bounded on $L^{p,w}$ if $w \in A_p$ and $1 < p < \infty$. Moreover, T is bounded from $L^{1,w}$ to $WL^{1,w}$ if $w \in A_1$.*

We now present the definition of the generalized weighted Morrey spaces and the generalized weighted weak Morrey spaces which will become the spaces of our interest in this article.

Definition 2.9. Let $1 \le p < \infty, w \in A_p$, and ψ be a non-negative function on $\mathbb{R}^n \times (0,\infty)$. The *generalized weighted Morrey space* $\mathcal{M}_\psi^{p,w} = \mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$ is the set of all functions $f \in L_{loc}^{p,w}$ such that

$$\begin{aligned}\|f\|_{\mathcal{M}_\psi^{p,w}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \left(\frac{1}{w(B(a,r))} \int_{B(a,r)} |f(x)|^p w(x)dx \right)^{1/p} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))} < \infty.\end{aligned}$$

Definition 2.10. Let $1 \le p < \infty, w \in A_p$, and ψ be a non-negative function defined on $\mathbb{R}^n \times (0,\infty)$. The *generalized weighted weak Morrey space* $W\mathcal{M}_\psi^{p,w} = W\mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$ is the set of all functions $f \in L_{loc}^{p,w}$ such that

$$\begin{aligned}\|f\|_{W\mathcal{M}_\psi^{p,w}} &= \sup_{a\in\mathbb{R}^n, r>0} \sup_{\gamma>0} \frac{1}{\psi(a,r)} \frac{\gamma}{w(B(a,r))^{\frac{1}{p}}} w(\{x \in \mathbb{R}^n : |f(x)| > \gamma\})^{\frac{1}{p}} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{WL^{p,w}(B(a,r))} < \infty.\end{aligned}$$

Note that if we set w to be constant a.e., then $\mathcal{M}_\psi^{p,w} = \mathcal{M}_\psi^p$ and $W\mathcal{M}_\psi^{p,w} = W\mathcal{M}_\psi^p$. Moreover, if we set $\psi(a,r) = |B(a,r)|^{-\frac{1}{q}}$ and w is constant a.e., then $\mathcal{M}_\psi^{p,w}$ is the classical Morrey space $\mathcal{M}_q^p$ and $W\mathcal{M}_\psi^{p,w}$ is the classical weak Morrey space $W\mathcal{M}_q^p$.

With Definitions 2.9 and 2.10, we shall investigate the boundedness of the classical operators: the Hardy-Littlewood maximal operator, the fractional integral operators, the fractional maximal operators, and the Calderón-Zygmund operators on those spaces in the next section.

We end this section with lemmas which will be used later in proving our main theorems.

Lemma 2.11. *Let φ be a non-negative function on $\mathbb{R}^n \times (0,\infty)$ such that the map $r \mapsto \varphi(a,r)$ is increasing for each $a \in \mathbb{R}^n$. Let $w \in A_p$ where $1 \le p < \infty$. Then, for every ball $B(a,r)$, we have*

$$\varphi(a,r) \le Cw(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}} \varphi(a,s)$$

and

$$\varphi(a,r) \le Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}} \varphi(a,s) \frac{ds}{s}$$

where $C > 0$ is independent of the function φ, $a \in \mathbb{R}^n$ and $r > 0$.

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$. By Theorem 2.3 and the fact that the $s \mapsto \varphi(a, s)$ map is increasing for each $a \in \mathbb{R}^n$, we have

$$\begin{aligned} w(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s) &= \sup_{r<s<\infty} \frac{w(B(a,r))^{\frac{1}{p}}}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s) \\ &\geq C \sup_{r<s<\infty} \frac{r^n}{s^n}\varphi(a,r) \\ &= C\varphi(a,r). \end{aligned}$$

Moreover,

$$\begin{aligned} w(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s} &= \int_r^\infty \frac{w(B(a,r))^{\frac{1}{p}}}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s} \\ &\geq \int_r^\infty C\frac{|B(a,r)|}{|B(a,s)|}\varphi(a,s)\frac{ds}{s} \\ &\geq C\int_r^{2r} \frac{r^n}{(2r)^n}\varphi(a,r)\frac{ds}{2r} \\ &= C\varphi(a,r). \end{aligned}$$

Therefore,

$$\varphi(a,r) \leq Cw(B(a,r))^{\frac{1}{p}} \sup_{r<s<\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)$$

and

$$\varphi(a,r) \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}}\varphi(a,s)\frac{ds}{s},$$

which proves the lemma. □

Lemma 2.12. *Let $1 \leq p < \infty$ and $w \in A_p$. For each $(a,s) \in \mathbb{R}^n \times (0,\infty)$, we have*

$$\frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy \leq \frac{C}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}, \quad f \in L^{p,w}_{loc}$$

where C is constant independent of a, s and f.

Proof. First, consider the case where $1 < p < \infty$. By using Hölder's inequality, we have for every $a \in \mathbb{R}^n$ and $s > 0$,

$$\begin{aligned} &\frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy \\ &= \frac{w(B(a,s))^{\frac{1}{p}}}{|B(a,s)|}\frac{1}{w(B(a,s))^{\frac{1}{p}}}\int_{B(a,s)} |f(y)|\frac{w(y)^{\frac{1}{p}}}{w(y)^{\frac{1}{p}}}dy \\ &\leq \frac{w(B(a,s))^{\frac{1}{p}}}{|B(a,s)|}\frac{1}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}\left(\int_{B(a,s)} w(y)^{-\frac{p'}{p}}dy\right)^{\frac{1}{p'}}. \end{aligned}$$

From the previous inequality and using the assumption that $w \in A_p$, we have the following inequality

$$\begin{aligned}
&\frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\\
&=\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)dy\right)^{\frac{1}{p}}\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)^{-\frac{p'}{p}}dy\right)^{\frac{1}{p'}}\\
&\quad\times\frac{1}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}\\
&\le\frac{C}{w(B(a,s))^{\frac{1}{p}}}\left(\int_{B(a,s)}|f(y)|^p w(y)dy\right)^{\frac{1}{p}}\\
&=\frac{C}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}.
\end{aligned}$$

This proves the case where $1 < p < \infty$.

For $p = 1$, we use Hölder's inequality and the assumption that $w \in A_1$ to get

$$\begin{aligned}
&\frac{1}{|B(a,s)|}\int_{B(a,s)}|f(y)|dy\\
&=\frac{w(B(a,s))}{|B(a,s)|}\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|\frac{w(y)}{w(y)}dy\\
&\le\frac{w(B(a,s))}{|B(a,s)|}\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\,\|w^{-1}\|_{L^\infty(B(a,s))}\\
&=\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)dy\,\|w^{-1}\|_{L^\infty(B(a,s))}\cdot\frac{1}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\\
&\le\frac{C}{w(B(a,s))}\int_{B(a,s)}|f(y)|w(y)dy\\
&=\frac{C}{w(B(a,s))}\|f\|_{L^{1,w}(B(a,s))},
\end{aligned}$$

as desired. □

Corollary 2.13. *For each $1 \le p < \infty$ and $w \in A_p$, there exists $C > 0$ such that for every $a \in \mathbb{R}^n$ and $r > 0$, we have*

$$\int_{\mathbb{R}^n\setminus B(a,r)}\frac{|f(y)|}{|a-y|^n}dy \le C\int_r^\infty w(B(a,s))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s},\quad f\in L^{p,w}_{loc}.$$

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$. Then, by Fubini's Theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(a,r)} \frac{f(y)}{|a-y|^n}dy &= \int_{B(a,r)^c} |f(y)| \int_{|a-y|}^{\infty} \frac{1}{s^n}\frac{ds}{s}dy \\ &= \int_r^{\infty}\int_{B(a,s)\setminus B(a,r)} |f(y)|dy\frac{1}{s^n}\frac{ds}{s} \\ &\leq C\int_r^{\infty} \frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy\frac{ds}{s}.\end{aligned}$$

Hence, it follows from Lemma 2.12 that

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(a,r)} \frac{|f(y)|}{|a-y|^n}dy &\leq C\int_r^{\infty} \frac{1}{|B(a,s)|}\int_{B(a,s)} |f(y)|dy\frac{ds}{s} \\ &\leq C\int_r^{\infty} \frac{1}{w(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s},\end{aligned}$$

as claimed. □

3. Hardy-Littlewood Maximal Operator on Generalized Weighted Morrey Spaces

In this section, we prove the boundedness of the Hardy-Littlewood operator M on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. Keeping in mind Theorem 2.6, we have the following results.

Theorem 3.1. *Let $1 \leq p < \infty, w \in A_p$. Then, for every $a \in \mathbb{R}^n$ and $r > 0$,*

$$\|Mf\|_{L^{p,w}(B(a,r))} \leq C_1 w(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)}, \quad f \in L^{p,w}_{loc},$$

for $1 < p < \infty$, and

$$\|Mf\|_{WL^{1,w}(B(a,r))} < C_2 w(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}B(a,t)}, \quad f \in L^{1,w}_{loc},$$

where C_1 and C_2 are constants that do not depend on $f, a,$ and r.

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$, and write f in the form of $f := f_1 + f_2$ where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. Assume that $1 < p < \infty$. Since $w \in A_p$, M is bounded on $L^{p,w}$. Thus,

$$\|Mf\|_{L^{p,w}(B(a,r))} \leq \|Mf_1\|_{L^{p,w}(B((a,r))} + \|Mf_2\|_{L^{p,w}(B((a,r))}$$

and

$$\|Mf_1\|_{L^{p,w}(B(a,r))} \leq \|Mf_1\|_{L^{p,w}} \leq C\|f_1\|_{L^{p,w}} \leq C\|f\|_{L^{p,w}(B(a,2r))}.$$

We can see that the map $r \mapsto \|f\|_{L^{p,w}(B(a,2r))}$ is increasing for each $a \in \mathbb{R}^n$. Then, by Theorem 2.3 and Lemma 2.11,

$$\|Mf_1\|_{L^{p,w}(B(a,r))} \leq Cw(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} \frac{1}{w(B(a,t))^{\frac{1}{p}}}\|f\|_{L^{p,w}(B(a,t))}.$$

Let $x \in B(a,r)$. If $y \in B(x,t) \cap B(a,2r)^c$, then $r = 2r - r \leq |y-a| - |a-x| \leq |y-x| < t$. In other words,

$$\int_{B(x,t)\cap B(a,2r)^c} |f(y)|dy = 0, \quad t \leq r.$$

Moreover, $|y-a| \le |y-x| + |x-a| \le t + r < 2t$. It then follows that

$$\begin{aligned}
&Mf_2(x)\\
&= \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f_2(y)|dy\\
&\le \max\left(\sup_{t>r} \frac{1}{|B(x,t)|} \int_{B(x,t)\cap B(a,2r)^c} |f(y)|dy, \sup_{0<t\le r} \frac{1}{|B(x,t)|} \int_{B(x,t)\cap B(a,2r)^c} |f(y)|dy \right)\\
&= \sup_{t>r} \frac{1}{|B(x,t)|} \int_{B(x,t)\cap B(a,2r)^c} |f(y)|dy\\
&\le \sup_{t>r} \frac{1}{|B(x,t)|} \int_{B(a,2t)} |f(y)|dy\\
&= C \sup_{t>2r} \frac{1}{|B(a,t)|} \int_{B(a,t)} |f(y)|dy\\
&\le C \sup_{t>r} \frac{1}{w(B(a,t))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,t))}.
\end{aligned}$$

and

$$Mf_2(x) \le C \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,t))}, \quad x \in B(a,r). \tag{3.1}$$

Hence,

$$\|Mf_2\|_{L^{p,w}(B(a,r))} \le Cw(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}} \|f\|_{L^{p,w}B(a,t)},$$

and so we conclude that

$$\|Mf\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}} \|f\|_{L^{p,w}B(a,t)}.$$

Assume now that $p = 1$. Thus,

$$\|Mf\|_{WL^{1,w}(B(a,r))} \le 2\left(\|Mf_1\|_{WL^{1,w}(B(a,r)} + \|Mf_2\|_{WL^{1,w}(B(a,r))} \right).$$

Since M is bounded from $L^{1,w}$ to $WL^{1,w}$, we have

$$\|Mf_1\|_{WL^{1,w}(B(a,r))} \le \|Mf_1\|_{WL^{1,w}} \le C\|f_1\|_{L^{1,w}} = C\|f\|_{L^{1,w}(B(a,2r))}.$$

Theorem 2.3 and Lemma 2.11 then imply that

$$\|Mf_1\|_{WL^{1,w}(B(a,r))} \le Cw(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1} \|f\|_{L^{1,w}B(a,t)}.$$

On the other hand, we can see that (3.1) also holds for $p = 1$ which implies the following estimates.

$$\begin{aligned}\|Mf_2\|_{WL^{1,w}(B(a,r))} &= \sup_{\gamma>0} \gamma w(\{x \in B(a,r) : |Mf_2(x)| > \gamma\}) \\ &= \sup_{\gamma>0} \gamma \int_{\{x\in B(a,r):|Mf_2(x)|>\gamma\}} w(x)dx \\ &\le \int_{B(a,r)} |Mf_2(x)|w(x)dx \\ &\le C \int_{B(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}(B(a,t))} w(x)dx \\ &= Cw(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{1,w}(B(a,t))}.\end{aligned}$$

Therefore,

$$\|Mf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \sup_{r<t<\infty} w(B(a,t))^{-1}\|f\|_{L^{p,w}(B(a,t))}$$

and this completes the proof of Theorem 3.1. □

The following theorem is one of our main results.

Theorem 3.2. *Let $1 \le p < \infty, w \in A_p$, and M be the Hardy-Littlewood maximal operator. Suppose that ψ_1 and ψ_2 are two positive functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\sup_{r<t<\infty} \psi_1(a,t) \le C\psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where C is a constant that does not depend on a and r. Then,

(1) *M is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1 < p < \infty$.*

(2) *M is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

Proof. First, assuming that $1 < p < \infty$, let $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 3.1 and the hypothesis about ψ_1 and ψ_2, we get

$$\begin{aligned}\|Mf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}}\|Mf\|_{L^{p,w}(B(a,r))} \\ &\le C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}B(a,t)} \\ &= C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \frac{\psi_1(a,t)}{\psi_1(a,t)} w(B(a,t))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,t))} \\ &\le C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t)\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \\ &= C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \\ &\le C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.\end{aligned}$$

Therefore, we conclude that M is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$.

Next, let $p = 1$ and $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 3.1 and the hypothesis concerning ψ_1 and ψ_2, we get

$$\begin{aligned}
\|Mf\|_{W\mathcal{M}^{1,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1} \|Mf\|_{WL^{1,w}(B(a,r))} \\
&\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} w(B(a,t))^{-1} \|f\|_{L^{1,w}B(a,t)} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \frac{\psi_1(a,t)}{\psi_1(a,t)} w(B(a,t))^{-1} \|f\|_{L^{1,w}(B(a,t))} \\
&\leq C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \\
&= C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \sup_{r<t<\infty} \psi_1(a,t) \\
&\leq C \|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that M is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$, and this completes the proof of Theorem 3.2. □

Let $a \in \mathbb{R}^n$. Consider the function $t \mapsto \psi_1(a,t)$ on $(0,\infty)$ by $\psi(a,t) = n$ where $t = 1/n$ for some $n \in \mathbb{N}$ and $\psi(a,t) = te^{-t}$ for otherwise. We also consider the function $t \mapsto \psi_2(a,t)$ by $\psi_2(a,t) = e^{-t}$ for every $t > 0$. We can see that

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \leq \psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, but there is no $C > 0$ such that

$$\sup_{r<t<\infty} \psi_1(a,t) \leq \psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$. Hence, we also investigate the condition

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \leq C\psi_2(a,r)$$

for the boundedness of the Hardy-Littlewood maximal operator and obtain the following results.

Theorem 3.3. *Let $1 \leq p < \infty$ and $w \in A_p$. Suppose that ψ_1 and ψ_2 are positive functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \psi_1(a,t) \frac{dt}{t} \leq C\psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$ where C is a constant that does not depend on a and r. Then,

(1) *M is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1 < p < \infty$.*

(2) *M is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

Before we present the proof of Theorem 3.3, we prove the following theorem.

Theorem 3.4. *Let $1 \le p < \infty$ and $w \in A_p$. Then, for every $a \in \mathbb{R}^n$ and $r > 0$,*

$$\|Mf\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}, \quad f \in L^{p,w}_{loc},$$

for $1 < p < \infty$, and

$$\|Mf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}, \quad f \in L^{1,w}_{loc},$$

where C_1 and C_2 are positive constants that are independent of f, a, and r.

Proof. Given $a \in \mathbb{R}^n$ and $r > 0$, we write $f := f_1 + f_2$ where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. Then, by Theorem 2.3, Theorem 2.6, and Lemma 2.11,

$$\begin{aligned}\|Mf_1\|_{L^{p,w}(B(a,r))} &\le C\|f\|_{L^{p,w}(B(a,r))} \\ &\le Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}\end{aligned}$$

for $1 < p < \infty$. Meanwhile, for $p = 1$, we have

$$\begin{aligned}\|Mf_1\|_{WL^{1,w}(B(a,r))} &\le C\|f\|_{L^{1,w}(B(a,r))} \\ &\le Cw(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.\end{aligned}$$

Regarding f_2, for every $x \in B(a,r)$, we have

$$Mf_2(x) \le C \sup_{t>r} \frac{1}{|B(a,2t)|} \int_{B(a,2t)} |f(y)|dy,$$

so that we get

$$Mf_2(x) \le C \sup_{t>2r} \frac{1}{|B(a,t)|} \int_{B(a,t)} |f(y)|dy.$$

Hence,

$$Mf_2(x) \le C \sup_{t>r} \frac{1}{w(B(a,t))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,t))}$$

for $1 \le p < \infty$. Therefore,

$$\begin{aligned}Mf_2(x) &\le C \sup_{t>r} \int_t^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\ &\le \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s},\end{aligned}$$

for every $x \in B(a,r)$. It thus follows that

$$\|Mf_2\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}$$

for $1 < p < \infty$ and

$$\|Mf_2\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

This proves Theorem 3.4. □

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. First, consider the case where $1 < p < \infty$. Given $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 3.4 and the assumption concerning ψ_1 and ψ_2, we get

$$\begin{aligned}
\|Mf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}} \|Mf\|_{L^{p,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that M is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$.

Now let $p = 1$ and suppose that $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 3.4 and the assumption concerning ψ_1 and ψ_2, we get

$$\begin{aligned}
\|Mf\|_{W\mathcal{M}^{1,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1} \|Mf\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that M is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$. This completes the proof of Theorem 3.3. □

4. Fractional Integral and Fractional Maximal Operators on Generalized Weighted Morrey Spaces

In this section, we prove the boundedness of the fractional integral operator I_α on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. The results then imply the boundedness of the fractional maximal operators on those spaces. As an application of Theorem 2.7, we have the following results.

Theorem 4.1. *Let $0<\alpha<n, 1\le p<n/\alpha, 1/q=1/p-\alpha/n$, and $w\in A_{p,q}$. Then,*

$$\|I_\alpha f\|_{L^{q,w^q}(B(a,r))}\le C_1 w^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s},$$

for every $a\in\mathbb{R}^n$ $r>0$, and $f\in L^{p,w}_{loc}$ where $1<p<\infty$, and

$$\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))}\le C_2 w^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s},$$

for every $a\in\mathbb{R}^n$ $r>0$, and $f\in L^{1,w}_{loc}$. C_1 and C_2 are constants that do not depend on f,a, and r.

Proof. Given $a\in\mathbb{R}^n$ and $r>0$, we decompose the function f as $f:=f_1+f_2$ where $f_1:=f\mathcal{X}_{B(a,2r)}$, so that

$$I_\alpha f(x)=I_\alpha f_1(x)+I_\alpha f_2(x).$$

Suppose that $1<p<n/\alpha$. By Theorem 2.7, I_α is bounded from L^{p,w^p} to L^{q,w^q}. Hence,

$$\|I_\alpha f_1\|_{L^{q,w^q}(B(a,r))}\le\|I_\alpha f_1\|_{L^{q,w^q}}\le C\|f_1\|_{L^{p,w^p}}=C\|f\|_{L^{p,w^p}(B(a,2r))}.$$

Since $w\in A_{p,q}$, it follows from Proposition 2.5 that $w^q\in A_q$. We see that the map $r\mapsto\|f\|_{L^{p,w^p}(B(a,2r))}$ is increasing for each $a\in\mathbb{R}^n$, and so by Theorem 2.3 and Lemma 2.11 we have

$$\|I_\alpha f_1\|_{L^{q,w^q}}\le Cw^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s}.$$

Next, we obtain the same estimate for $I_\alpha f_2$. For this, we observe that

$$|I_\alpha f_2(x)|\le\int_{B(a,2r)^c}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy.$$

The inequalities $|a-x|<r$ and $|x-y|\ge 2r$ implies

$$\frac{1}{2}|a-y|\le|x-y|\le\frac{3}{2}|a-y|.$$

Then,

$$|I_\alpha f_2(x)|\le C\int_{B(a,2r)^c}\frac{|f(y)|}{|a-y|^{n-\alpha}}dy,\quad x\in B(a,r).$$

By Fubini's theorem,

$$\begin{aligned}|I_\alpha f_2(x)|&\le C\int_{B(a,2r)^c}\frac{|f(y)|}{|a-y|^{n-\alpha}}dy\\&=C\int_{B(a,2r)^c}|f(y)|\int_{|a-y|}^\infty\frac{1}{s^{n-\alpha}}\frac{ds}{s}dy\\&=C\int_r^\infty\int_{B(a,s)\setminus B(a,r)}\frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}}|f(y)|dy\frac{ds}{s}\\&=\int_r^\infty\frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}}\int_{B(a,s)}|f(y)|dy\frac{ds}{s}.\end{aligned}$$

Next, by Hölder's inequality and the assumption that $w \in A_{p,q}$, we have

$$\begin{aligned}
&\frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}} \int_{B(a,s)} |f(y)|dy \\
&= \frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}} \int_{B(a,s)} \frac{|f(y)|w(y)}{w(y)}dy \\
&\le \frac{1}{|B(a,s)|^{1+\frac{1}{q}-\frac{1}{p}}} \left(\int_{B(a,s)} |f(y)|^p w(y)^p dy\right)^{\frac{1}{p}} \left(\int_{B(a,s)} w(y)^{-p'} dy\right)^{\frac{1}{p'}} \\
&= w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \left(\frac{1}{|B(a,s)|} \int_{B(a,s)} w(y)^q dy\right)^{\frac{1}{q}} \\
&\quad \times \left(\frac{1}{|B(a,s)|} \int_{B(a,s)} w(y)^{-p'} dy\right)^{\frac{1}{p'}} \\
&\le C w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))}.
\end{aligned}$$

Hence,

$$|I_\alpha f_2(x)| \le C \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s}, \quad x \in B(a,r),$$

and this implies that

$$\|I_\alpha f_2\|_{L^{q,w^q}(B(a,r))} \le C w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s}.$$

Therefore,

$$\|I_\alpha f\|_{L^{q,w^q}(B(a,r))} \le C_1 w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s}.$$

Next, we prove the case where $p = 1$. Note that

$$\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))} \le 2 \left(\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))} + \|I_\alpha f_2\|_{WL^{q,w^q}(B(a,r))}\right).$$

By Theorem 2.7, we have

$$\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))} \le \|I_\alpha f_1\|_{WL^{q,w^q}} \le C\|I_\alpha f_1\|_{L^{1,w}} = C\|I_\alpha f\|_{L^{1,w}(B(a,2r))}.$$

Since $w \in A_{1,q}$, it follows from Proposition 2.5 that $w^q \in A_q$. By the same argument as for the case $p > 1$, we obtain

$$\|I_\alpha f_1\|_{WL^{q,w^q}(B(a,r))} \le C w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

As for f_2, we have

$$|I_\alpha f_2(x)| \le C \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^{n-\alpha}} dy, \quad x \in B(a,r),$$

and so, by Fubini's theorem, we obtain

$$\begin{aligned}|I_\alpha f_2(x)| &\leq C\int_{\mathbb{R}^n\setminus B(a,2r)} \frac{|f(y)|}{|a-y|^{n-\alpha}}dy \\ &= C\int_{B(a,2r)^c}|f(y)|\int_{|a-y|}^{\infty}\frac{1}{s^{n-\alpha}}\frac{ds}{s}dy \\ &= C\int_r^\infty\int_{B(a,s)\setminus B(a,r)}\frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}}|f(y)|dy\frac{ds}{s} \\ &= \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{q}}}\int_{B(a,s)}|f(y)|dy\frac{ds}{s}.\end{aligned}$$

By Hölder's inequality combined with the assumption $w \in A_{1,q}$ and the fact that $q > 1$, we get

$$\begin{aligned}&\frac{1}{|B(a,s)|^{\frac{1}{q}}}\int_{B(a,s)}|f(y)|dy \\ &\leq \frac{1}{|B(a,s)|^{\frac{1}{q}}}\int_{B(a,s)}|f(y)|\frac{w(y)}{w(y)}dy \\ &\leq \frac{1}{|B(a,s)|^{\frac{1}{q}}}\left(\int_{B(a,s)}|f(y)|w(y)dy\right)\|w^{-1}\|_{L^\infty(B(a,s))} \\ &= w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,s))}\left(\frac{1}{|B(a,s)|}\int_{B(a,s)}w(y)^q dy\right)^{\frac{1}{q}}\|w^{-1}\|_{L^\infty(B(a,s))} \\ &\leq Cw^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,s))}.\end{aligned}$$

Hence, for $x \in B(a,r)$,

$$\begin{aligned}|I_\alpha f_2(x)| &\leq C\int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{q}}}\int_{B(a,s)}|f(y)|dy\frac{ds}{s} \\ &\leq C\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s},\end{aligned}$$

and

$$\begin{aligned}\|I_\alpha f_2\|_{WL^{q,w^q}(B(a,r))} &= \sup_{\gamma>0}\gamma w^q(\{x\in B(a,r) : |I_\alpha f_2(x)| > \gamma\})^{\frac{1}{q}} \\ &= \sup_{\gamma>0}\gamma\left(\int_{\{x\in B(a,r):|I_\alpha f_2(x)|>\gamma\}} w(x)^q dx\right)^{\frac{1}{q}} \\ &\leq \sup_{\gamma>0}\left(\int_{\{x\in B(a,r):|I_\alpha f_2(x)|>\gamma\}}|I_\alpha f_2(y)|^q w(y)^q dy\right)^{\frac{1}{q}} \\ &= \left(\int_{B(a,r)}|I_\alpha f_2(y)|^q w(y)^q dy\right)^{\frac{1}{q}} \\ &\leq Cw^q(B(a,r))^{\frac{1}{q}}\int_r^\infty w^q(B(a,s))^{-\frac{1}{q}}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s}.\end{aligned}$$

Therefore,

$$\|I_\alpha f\|_{WL^{q,w^q}(B(a,r))} \le C_2 w^q(B(a,r))^{\frac{1}{q}} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

This completes the proof of Theorem 4.1. □

The following theorem is our main results concerning the boundedness of the fractional integrals on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

Theorem 4.2. *Let $0 < \alpha < n, 1 \le p < n/\alpha, 1/q = 1/p - \alpha/n, w \in A_{p,q}$, and I_α be the fractional integral operator. Suppose that ψ_1 and ψ_2 are non-negative functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \frac{w^p(B(a,t))^{\frac{1}{p}}}{w^q(B(a,t))^{\frac{1}{q}}} \psi_1(a,t) \frac{dt}{t} \le C\psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$, where C is a constant that does not depend on a and r. Then,

(1) *I_α is bounded from $\mathcal{M}_{\psi_1}^{p,w^p}$ to $\mathcal{M}_{\psi_2}^{q,w^q}$ for $1 < p < \infty$.*
(2) *I_α is bounded from $\mathcal{M}_{\psi_1}^{1,w}$ to $W\mathcal{M}_{\psi_2}^{q,w^q}$.*

Proof. First, we prove the assertion for $1 < p < n/\alpha$. Let $f \in \mathcal{M}_{\psi_1}^{p,w^p}$. By using Theorem 4.1 and the assumption on ψ_1 and ψ_2, we get

$$\begin{aligned}
\|I_\alpha f\|_{\mathcal{M}_{\psi_2}^{q,w^q}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \left(\frac{1}{w^q(B(a,r))} \int_{B(a,r)} |I_\alpha f(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w^q(B(a,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L^{q,w^q}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{w^q(B(a,s))^{-\frac{1}{q}}}{w^p(B(a,s))^{-\frac{1}{p}}} \|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_t^\infty \psi_1(a,s) \frac{w^p(B(a,s))^{\frac{1}{p}}}{w^q(B(a,s))^{\frac{1}{q}}} \frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}_{\psi_1}^{p,w^p}}.
\end{aligned}$$

Therefore, we conclude that I_α is bounded from $\mathcal{M}_{\psi_1}^{p,w^p}$ to $\mathcal{M}_{\psi_2}^{q,w^q}$.

Next, let $p = 1$ and let $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 4.1 and the assumption on ψ_1 and ψ_2, we get

$$\begin{aligned}
\|I_\alpha f\|_{W\mathcal{M}^{q,w^q}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w^q(B(a,r))^{-\frac{1}{q}} \|I_\alpha f\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w^q(B(a,s))^{-\frac{1}{q}} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{w^q(B(a,s))^{-\frac{1}{q}}}{w(B(a,s))^{-1}} \|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_t^\infty \psi_1(a,s) \frac{w(B(a,s))}{w^q(B(a,s))^{\frac{1}{q}}} \frac{ds}{s} \le C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that I_α is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$. This completes the proof of Theorem 4.2. □

By the relation (1.1) and Theorem 4.2, we have the following corollary for the fractional maximal operator M_α.

Corollary 4.3. *Let $0 < \alpha < n, 1 \le p < n/\alpha, 1/q = 1/p - \alpha/n$, and $w \in A_{p,q}$. Suppose that ψ_1 and ψ_2 are non-negative functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \frac{w^p(B(a,t))^{\frac{1}{p}}}{w^q(B(a,s))^{\frac{1}{q}}} \psi_1(a,s) \frac{dt}{t} \le C\psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$ where C is a constant that does not depend on a and r. Then,

(1) *M_α is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$ for $1 < p < \infty$.*
(2) *M_α is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{q,w^q}_{\psi_2}$.*

5. Calderón-Zygmund Operators on Generalized Weighted Morrey Spaces

In this section, we prove the boundedness of the Calderón-Zygmund operators $T = T_K$ on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces. As stated earlier, we have Theorem 2.8 about the boundedness of the Calderón-Zygmund operators on weighted Lebesgue spaces and weighted weak Lebesgue spaces. By using this theorem, we have the following results.

Theorem 5.1. *Let $1 \le p < \infty, w \in A_p$ and $f \in L^{p,w}_{loc}$. Then, for every $a \in \mathbb{R}^n$ and $r > 0$*

$$\|Tf\|_{L^{p,w}(B(a,r))} \le C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}$$

where $1 < p < \infty$, and

$$\|Tf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s},$$

where C_1 and C_2 are constants that do not depend on $f, a,$ and r.

Proof. Write f as $f := f_1 + f_2$ where $f_1 := f \cdot \mathcal{X}_{B(a,2r)}$. First, we consider the case where $1 < p < \infty$. Then, since $w \in A_p$, we know that T is bounded on $L^{p,w}$. Thus, for every $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\|Tf\|_{L^{p,w}(B(a,r))} \leq \|Tf_1\|_{L^{p,w}(B((a,r))} + \|Tf_2\|_{L^{p,w}(B((a,r))}$$

and

$$\|Tf_1\|_{L^{p,w}(B(a,r))} \leq \|Tf_1\|_{L^{p,w}} \leq C\|f_1\|_{L^{p,w}} \leq C\|f\|_{L^{p,w}(B(a,2r))}.$$

By Theorem 2.3 and Lemma 2.11,

$$\|Tf_1\|_{L^{p,w}(B(a,r))} \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Next, we note that for $x \in B(a,r)$, we have

$$|Tf_2(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|x-y|^n} dy.$$

On other hand, the inequalities $|a-x| < r$ and $|x-y| \geq 2r$ imply that

$$\frac{1}{2}|a-y| \leq |x-y| \leq \frac{3}{2}|a-y|.$$

Then, by using Lemma 2.12,

$$|Tf_2(x)| \leq C \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^n} dy \leq C \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Hence,

$$\|Tf_2\|_{L^{p,w}(B(a,r))} \leq Cw(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Therefore, we conclude that

$$\|Tf\|_{L^{p,w}(B(a,r))} \leq C_1 w(B(a,r))^{\frac{1}{p}} \int_r^\infty w(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w}(B(a,s))} \frac{ds}{s}.$$

Next, for the case where $p = 1$, we have

$$\|Tf\|_{WL^{1,w}(B(a,r))} \leq 2(\|Tf_1\|_{WL^{1,w}(B(a,r))} + \|Tf_2\|_{WL^{1,w}(B(a,r))})$$

and

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq \|Tf_1\|_{L^{1,w}} \leq C\|f_1\|_{L^{1,w}} \leq C\|f\|_{L^{1,w}(B(a,2r))}$$

for every $a \in \mathbb{R}^n$ and $r > 0$. By the boundedness of T from $L^{1,w}$ to $WL^{1,w}$, we have

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq \|Tf_1\|_{WL^{1,w}} \leq C\|f_1\|_{L^{1,w}} \leq C\|f\|_{L^{1,w}(B(a,2r))}.$$

By Theorem 2.3 and Lemma 2.11,

$$\|Tf_1\|_{WL^{1,w}(B(a,r))} \leq Cw(B(a,r)) \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

For f_2, we observe that for every $x \in B(a,r)$ we have

$$|Tf_2(x)| \leq C \int_r^\infty w(B(a,s))^{-1} \|f\|_{L^{1,w}(B(a,s))} \frac{ds}{s}.$$

Thus,

$$\begin{aligned}
\|Tf_2\|_{WL^{1,w}(B(a,r))} &= \sup_{\gamma>0} \gamma w(\{x \in B(a,r) : |Tf_2(x)| > \gamma\}) \\
&= \sup_{\gamma>0} \gamma \int_{\{x\in B(a,r):|Tf_2(x)|>\gamma\}} w(x)dx \\
&\le \int_{B(a,r)} |Tf_2(x)|w(x)dx \\
&\le Cw(B(a,r)) \int_r^\infty w(B(a,s))^{-1}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s}.
\end{aligned}$$

Therefore,

$$\|Tf\|_{WL^{1,w}(B(a,r))} \le C_2 w(B(a,r)) \int_r^\infty w(B(a,s))^{-1}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s},$$

and this proves Theorem 5.1. □

The following theorem is our main result concerning the boundedness of the Calderón-Zygmund operators on generalized weighted Morrey spaces and generalized weighted weak Morrey spaces.

Theorem 5.2. *Let $1 \le p < \infty, w \in A_p$, and T be the Calderón-Zygmund operator. Suppose that ψ_1 and ψ_2 are functions on $\mathbb{R}^n \times (0,\infty)$ satisfying*

$$\int_r^\infty \psi_1(a,t)\frac{dt}{t} \le C\psi_2(a,r)$$

for every $(a,r) \in \mathbb{R}^n \times (0,\infty)$ where C is a constant that does not depend on a and r. Then,

1. *T is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$ for $1 < p < \infty$.*
2. *T is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$.*

Proof. First, suppose that $1 < p < \infty$ and $f \in \mathcal{M}^{p,w}_{\psi_1}$. By using Theorem 5.1 and the assumption concerning ψ_1 and ψ_2, we get

$$\begin{aligned}
\|Tf\|_{\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-\frac{1}{p}}\|Tf\|_{L^{p,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s} \\
&= C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \frac{\psi_1(a,s)}{\psi_1(a,s)} w(B(a,s))^{-\frac{1}{p}}\|f\|_{L^{p,w}(B(a,s))}\frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s)\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}\frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n,r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s)\frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}^{p,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that T is bounded from $\mathcal{M}^{p,w}_{\psi_1}$ to $\mathcal{M}^{p,w}_{\psi_2}$.

Next, let $p = 1$ and $f \in \mathcal{M}^{1,w}_{\psi_1}$. By using Theorem 5.1 and the assumption on ψ_1 and ψ_2, we get

$$\begin{aligned}
\|Tf\|_{W\mathcal{M}^{p,w}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} w(B(a,r))^{-1}\|Tf\|_{WL^{1,w}(B(a,r))} \\
&\le C \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty w(B(a,s))^{-1}\|f\|_{L^{1,w}(B(a,s))}\frac{ds}{s} \\
&\le C \sup_{a\in\mathbb{R}^n} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s)\|f\|_{\mathcal{M}^{1,w}_{\psi_1}}\frac{ds}{s} \\
&= C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}} \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s)\frac{ds}{s} \\
&\le C\|f\|_{\mathcal{M}^{1,w}_{\psi_1}}.
\end{aligned}$$

Therefore, we conclude that T is bounded from $\mathcal{M}^{1,w}_{\psi_1}$ to $W\mathcal{M}^{1,w}_{\psi_2}$ and this completes the proof. □

Remark 5.3. By using the results in [19], we can extend Theorem 5.2 by replacing T with θ-type Calderón-Zygmund operators T_θ. The definition of θ-type Calderón-Zygmund Operator T_θ may be found in [24]. Accordingly one can obtain a result that is more general than [23].

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