Research Article

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On inclusion properties of discrete Morrey spaces

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Abstract: We discuss a necessary condition for inclusion relations of weak type discrete Morrey spaces which can be seen as an extension of the results in [H. Gunawan, E. Kikianty and C. Schwanke, Discrete Morrey spaces and their inclusion properties, *Math. Nachr.* **291** (2018), no. 8–9, 1283–1296] and [D. D. Haroske and L. Skrzypczak, Morrey sequence spaces: Pitt's theorem and compact embeddings, *Constr. Approx.* **51** (2020), no. 3, 505–535]. We also prove a proper inclusion from weak type discrete Morrey spaces into discrete Morrey spaces. In addition, we give a necessary condition for this inclusion. Some connections between the inclusion properties of discrete Morrey spaces and those of Morrey spaces are also discussed.

Keywords: Discrete Morrey spaces, inclusion properties, Morrey spaces

MSC 2010: 42B35, 46A45, 46B45

1 Introduction

Let $1 \le p \le q < \infty$. The discrete Morrey space $\ell_q^p = \ell_q^p(\mathbb{Z})$ introduced in [5] is defined to be the set of all sequences $x = (x_j)_{j \in \mathbb{Z}}$ such that

$$\sup_{m \in \mathbb{Z},\, N \in \mathbb{N}_0} |S_{m,N}|^{\frac{p}{q}-1} \sum_{j \in S_{m,N}} |x_j|^p < \infty,$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $S_{m,N} := \{m-N, \dots, m, \dots, m+N\}$, and $|S_{m,N}| = 2N+1$. This space is a Banach space with the norm

$$\|x\|_{\ell_q^p} := \sup_{m \in \mathbb{Z},\, N \in \mathbb{N}_0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \Big(\sum_{j \in S_{m,N}} |x_j|^p \Big)^{\frac{1}{p}}.$$

We remark that, for $p = q$, we have $\ell_p^p = \ell^p$.

Moreover, it is shown in [5] that ℓ^p is a proper subset of ℓ_q^p whenever $1 \le p < q < \infty$. In the same paper, the authors also prove that if $1 \le p_1 \le p_2 \le q$, then

$$\ell_q^{p_2} \subseteq \ell_q^{p_1}. \tag{1.1}$$

In addition, it is shown that

$$\ell_{q_2}^{p_2} \subseteq \ell_{q_1}^{p_1} \implies q_2 \le q_1$$

whenever $1 \le p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, and $p_1 \le p_2$. Recently, these inclusion results have been extended to discrete Morrey spaces on $\mathbb{Z}^d$ in [8]. Moreover, their result includes the following theorem.

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Theorem 1.1 ([8, Theorem 3.1]). *Let $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$. Then the inclusion $\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$ holds if and only if $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$.*

In addition, the authors in [8] prove that the embedding (1.1) is never compact.

Many operators, such as a Hardy–Littlewood maximal operator and fractional integral operators, which were initially studied on $L^p(\mathbb{R}^d)$, have discrete analogues on $\ell^p(\mathbb{Z}^d)$ and their boundedness on $\ell^p(\mathbb{Z}^d)$ has been investigated, for example, in [14, 15], while the boundedness of "continuous" operators on "continuous" Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^d)$ has also been studied by quite a number of authors (see, for example, [1, 10]). It is therefore natural to inquire about the properties of the discrete analogues of those operators on discrete Morrey spaces, as is done in [6]. At the same time, it is also important to understand the properties of discrete Morrey spaces as well as we understand the "continuous" Morrey spaces.

The aim of the present paper is to extend the result on the inclusion of discrete Morrey spaces to their weak type and to investigate the relation between the inclusion properties of discrete Morrey spaces and those of Morrey spaces. The inclusion properties of weak type discrete Morrey spaces are discussed in Section 2.1. The proper inclusion relation between a discrete Morrey space and its weak type is given in Section 2.2. In Section 3, we discuss the relation between discrete Morrey spaces and "continuous" Morrey spaces on $\mathbb{R}$. Recall that, for $1 \le p \le q < \infty$, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^d)$ is defined to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ such that the norm

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \Big(\int_{B(a,r)} |f(y)|^p \, dy \Big)^{\frac{1}{p}}$$

is finite. We apply the result in [9] to recover the inclusion properties of discrete Morrey spaces from those of Morrey spaces. We also reprove a certain necessary condition for the inclusion relations between Morrey spaces in [2] by combining the results in Section 2 and those in [9]. Our main result in this section is a necessary condition for the proper inclusion relation between weak type discrete Morrey spaces and discrete Morrey spaces.

Let us refer to some previous works related to this paper. The inclusion properties of discrete Morrey spaces, their weak type spaces and their generalization are initially studied in [5]. Analogous results on Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^d)$ can be found in [2, 3, 7, 11–13].

Throughout this paper, we denote by C a positive constant which is independent of the sequence $x = (x_j)_{j\in\mathbb{Z}}$ and its value may be different from line to line. We write $A \lesssim B$ if there exists a positive constant C such that $A \le CB$, while $A \gtrsim B$ means $B \lesssim A$. In addition, we denote $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2 Main results

2.1 Inclusion property of weak type discrete Morrey spaces

Definition 2.1. Let $1 \le p \le q < \infty$. The weak type discrete Morrey space $w\ell^p_q$ is defined to be the set of all sequences $x = (x_j)_{j\in\mathbb{Z}}$ for which the quasi-norm

$$\|x\|_{w\ell^p_q} := \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0,\, \gamma>0} \gamma |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} |\{j \in S_{m,N} : |x_j| > \gamma\}|^{\frac{1}{p}}$$

is finite.

The quasi-norm in Definition 2.1 can be rewritten as

$$\|x\|_{w\ell^p_q} = \sup_{\gamma>0} \|\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}\|_{\ell^p_q},$$

where $1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}} := (1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}(j))_{j\in\mathbb{Z}}$ and

$$1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}(j) := \begin{cases} 1, & |x_j| > \gamma, \\ 0, & |x_j| \le \gamma. \end{cases}$$

Note that, for $p = q$, the space $w\ell^p_p$ is a weak type ℓ^p space. It is shown in [5, Example 3.1] that the space ℓ^p is a proper subset of $w\ell^p_p$. More general, the discrete Morrey space ℓ^p_q is a subset of $w\ell^p_q$ (see [5, Theorem 3.2]).

For the inclusion between weak type discrete Morrey spaces, it is proved in [5] that $w\ell^{p_2}_q \subseteq w\ell^{p_1}_q$ whenever $1 \le p_1 \le p_2 \le q < \infty$. Now we show that $p_1 \le p_2$ is a necessary condition for such inclusions.

Theorem 2.2. *Let $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$. Then $w\ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1}$ if and only if $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$.*

Proof. Suppose that $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$. Let $x \in w\ell^{p_2}_{q_2}$. Then, for every $\gamma > 0$, we have

$$\|\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}\|_{\ell^{p_2}_{q_2}} \le \|x\|_{w\ell^{p_2}_{q_2}} < \infty. \tag{2.1}$$

Therefore, $\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}} \in \ell^{p_2}_{q_2}$. According to Theorem 1.1, we have $\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}} \in \ell^{p_1}_{q_1}$ with

$$\|\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}\|_{\ell^{p_1}_{q_1}} \le \|\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}\|_{\ell^{p_2}_{q_2}}. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$\|\gamma 1_{\{j\in\mathbb{Z}:|x_j|>\gamma\}}\|_{\ell^{p_1}_{q_1}} \le \|x\|_{w\ell^{p_2}_{q_2}},$$

so $x \in w\ell^{p_1}_{q_1}$ and $\|x\|_{w\ell^{p_1}_{q_1}} \le \|x\|_{w\ell^{p_2}_{q_2}}$. This shows that

$$w\ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1}$$

with $\|\cdot\|_{w\ell^{p_1}_{q_1}} \le \|\cdot\|_{w\ell^{p_2}_{q_2}}$.

Now we prove the necessary condition for the inclusion $w\ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1}$. Let $x = (x_j)_{j\in\mathbb{Z}}$ be defined by

$$x_j := \begin{cases} 1, & |j| \le K, \\ 0, & |j| > K. \end{cases}$$

Then

$$(2K+1)^{\frac{1}{q_1}} = \|x\|_{\ell^{p_1}_{q_1}} = \|x\|_{w\ell^{p_1}_{q_1}} \le \|x\|_{w\ell^{p_2}_{q_2}} = \|x\|_{\ell^{p_2}_{q_2}} = (2K+1)^{\frac{1}{q_2}}.$$

Therefore, $(2K+1)^{\frac{1}{q_1}-\frac{1}{q_2}} \le 1$. Since $2K+1 \ge 1$, we have $\frac{1}{q_1} - \frac{1}{q_2} \le 0$, so that $q_2 \le q_1$. Assume to the contrary that $\frac{p_1}{q_1} > \frac{p_2}{q_2}$. Choose $v, w \in \mathbb{N}$ such that

$$\Big(\frac{q_1}{p_1} - 1\Big)w + \frac{2q_1}{p_1} < v < \Big(\frac{q_2}{p_2} - 1\Big)w + 2. \tag{2.3}$$

Let k_0 be a smallest positive integer such that $1 - \frac{1}{2^{2k_0}} > \frac{1}{2^{v+w-1}}$. Define $x = (x_j)_{j\in\mathbb{Z}}$ by

$$x_j := \begin{cases} 1, & |j| = 0, 1, 2, \dots, 2^{v+w}, \\ 1, & |j| = 2^{k(v+w)}, 2^{k(v+w)} - 2^{kw}, 2^{k(v+w)} - 2(2^{kw}), 2^{k(v+w)} - 3(2^{kw}), \\ & \qquad \dots, 2^{k(v+w)} - (2^{k(v-2)})(2^{kw}), \text{ where } k \in \mathbb{N} \cap [k_0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

We shall show that $x \in \ell^{p_2}_q$. For every $n \in \mathbb{N} \cap [k_0, \infty)$, define

$$m_n := \pm(2^{n(v+w)} - 2^{n(v+w-2)-1})$$

and let $N_n \in \{0, 1, 2, \dots, 2^{n(v+w-2)-1}\}$. Observe that, for every $m \in \mathbb{Z}$ with $2^{n(v+w)} - 2^{n(v+w)-2} \le |m| \le 2^{n(v+w)}$, we have

$$|S_{m,N_n}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N_n}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \le |S_{m_n,N_n}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m_n,N_n}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}}$$

We define

$$\ell_n := \max\{\ell \in \{0, 1, 2, \dots, 2^{n(v-2)-1}\} : 2^{nw}\ell \le N_n\}.$$

The inequalities $2^{nw}\ell_n \le N_n$ and $\frac{1}{q_2} - \frac{1}{p_2} \le 0$ imply

$$\begin{aligned} |S_{m_n,N_n}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m_n,N_n}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} &= (2N_n+1)^{\frac{1}{q_2}-\frac{1}{p_2}}(1+2\ell_n)^{\frac{1}{p_2}} \\ &\lesssim (2^{nw}\ell_n)^{\frac{1}{q_2}-\frac{1}{p_2}}(4\ell_n)^{\frac{1}{p_2}} \\ &= 4^{\frac{1}{p_2}}2^{\frac{nw}{q_2}-\frac{nw}{p_2}}(\ell_n)^{\frac{1}{q_2}} \lesssim 2^{n\left(\frac{v+w-2}{q_2}-\frac{w}{p_2}\right)}. \end{aligned}$$

Since $\frac{(v+w-2)}{q_2} - \frac{w}{p_2} < 0$, for every $m \in \mathbb{Z}$ with $2^{n(v+w)} - 2^{n(v+w)-2} \le |m| \le 2^{n(v+w)}$, we have

$$|S_{m,N_n}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N_n}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \lesssim 1. \tag{2.4}$$

If $N \le 2^{k_0(v+w)}$ and $m \in \mathbb{Z}$ with $|m| < 2^{k_0(v+w)} - 2^{k_0(v+w-2)}$, then

$$|S_{m,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \le |S_{0,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{0,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \le \Big(\sum_{j\in S_{0,2^{k_0(v+w)}}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} = C2^{\frac{k_0(v-2)}{p_2}}.$$

On the other hand, if $N > 2^{k_0(v+w)}$, then there exists $n \in \mathbb{N}$ such that

$$2^{n(v+w)} < N < 2^{(n+1)(v+w)}.$$

Hence, for every $N > 2^{k_0(v+w)}$ and $m \in \mathbb{Z}$ with $|m| < 2^{k_0(v+w)} - 2^{k_0(v+w-2)}$ we obtain

$$\begin{aligned}
|S_{m,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} &\le |S_{0,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{0,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \\
&\le |S_{0,2^{n(v+w)}}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{0,2^{(n+1)(v+w)}}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \\
&\le C(2^{n(v+w)}+1)^{\frac{1}{q_2}-\frac{1}{p_2}}\, 2^{\frac{(n+1)(v-2)}{p_2}} \\
&\le C2^{n\left(\frac{w}{q_2}-\frac{w}{p_2}+\frac{v}{q_2}-\frac{2}{p_2}\right)}.
\end{aligned} \tag{2.5}$$

Note that (2.3) implies $\frac{w}{q_2} - \frac{w}{p_2} + \frac{v}{q_2} - \frac{2}{p_2} < 0$. As a consequence of this inequality and (2.5), we have

$$|S_{m,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \lesssim 1 \tag{2.6}$$

for every $m \in \mathbb{Z}$ with $|m| < 2^{k_0(v+w)} - 2^{k_0(v+w-2)}$. Combining (2.4) and (2.6), we get

$$\sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q_2}-\frac{1}{p_2}}\Big(\sum_{j\in S_{m,N}} |x_j|^{p_2}\Big)^{\frac{1}{p_2}} \lesssim 1,$$

which means that $\|x\|_{\ell^{p_2}_{q_2}} < \infty$. Therefore, $x \in \ell^{p_2}_{q_2}$. Since $x \in \ell^{p_2}_{q_2}$ and $\ell^{p_2}_{q_2} \subseteq w\ell^{p_2}_{q_2}$, we have $x \in w\ell^{p_2}_{q_2}$. If we can prove that $x \notin w\ell^{p_1}_{q_1}$, then we obtain a contradiction, so we must have $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$. Now we show that $x \notin w\ell^{p_1}_{q_1}$. Observe that

$$\|x\|_{w\ell^{p_1}_{q_1}} \ge \frac{1}{2} \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\left|\left\{j \in S_{m,N} : |x_j| > \frac{1}{2}\right\}\right|^{\frac{1}{p_1}}.$$

Since either $x_j = 0$ or $x_j = 1$, we have

$$\begin{aligned}
\sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\left|\left\{j \in S_{m,N} : |x_j| > \frac{1}{2}\right\}\right|^{\frac{1}{p_1}} &= \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\Big(\sum_{j\in S_{m,N}:|x_j|>\frac{1}{2}} |x_j|^{p_1}\Big)^{\frac{1}{p_1}} \\
&= \sup_{m\in\mathbb{Z},\, N\in\mathbb{N}_0} |S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\Big(\sum_{j\in S_{m,N}} |x_j|^{p_1}\Big)^{\frac{1}{p_1}} \\
&= \|x\|_{\ell^{p_1}_{q_1}}.
\end{aligned}$$

Consequently,

$$\|x\|_{w\ell^{p_1}_{q_1}} \ge \frac{1}{2}\|x\|_{\ell^{p_1}_{q_1}}. \tag{2.7}$$

Let $n \in \mathbb{N} \cap [k_0, \infty)$. Observe that, for every $1 \le p < \infty$ we have

$$\sum_{|i|\le 2^{n(v+w)}} |x_i|^p = \Big(1 + 2^{v+w+1} + 2\sum_{k=k_0}^{n} (1 + 2^{k(v-2)})\Big) \sim 2^{n(v-2)}.$$

Therefore, for $p = p_1$, we have

$$|S_{0,2^{n(v+w)}}|^{\frac{1}{q_1}-\frac{1}{p_1}}\Big(\sum_{j\in S_{0,2^{n(v+w)}}}|x_j|^{p_1}\Big)^{\frac{1}{p_1}} \sim (2\cdot 2^{n(v+w)}+1)^{\frac{1}{q_1}-\frac{1}{p_1}}2^{\frac{n(v-2)}{p_1}} \geq (3\cdot 2^{n(v+w)})^{\frac{1}{q_1}-\frac{1}{p_1}}2^{\frac{n(v-2)}{p_1}}$$
$$= 3^{\frac{1}{q_1}-\frac{1}{p_1}}2^{n\left(\frac{v+w}{q}-\frac{w}{p_1}-\frac{2}{p_1}\right)}. \tag{2.8}$$

According to (2.3), we see that $\frac{(v+w)}{q_1}-\frac{w}{p_1}-\frac{2}{p_1}>0$, so that $2^{n\left(\frac{(v+w)}{q_1}-\frac{w}{p_1}-\frac{2}{p_1}\right)}\to\infty$ as $n\to\infty$. Combining this with (2.8), we have

$$\|x\|_{\ell^{p_1}_{q_1}} \gtrsim \sup_{n\in\mathbb{N}} 2^{n\left(\frac{(v+w)}{q_1}-\frac{w}{p_1}-\frac{2}{p_1}\right)}=\infty.$$

This inequality and (2.7) imply $x\notin w\ell^{p_1}_{q_1}$, as desired. □

As a consequence of Theorem 2.2, we have the following corollaries.

Corollary 2.3. *Let $1\leq p_1\leq q<\infty$ and $1\leq p_2\leq q<\infty$. Then $w\ell^{p_2}_q\subseteq w\ell^{p_1}_q$ if and only if $p_1\leq p_2$.*

Corollary 2.4. *Let $1\leq p\leq q_1<\infty$ and $1\leq p\leq q_2<\infty$. Then one has $q_2\leq q_1$ if and only if $w\ell^p_{q_2}\subseteq w\ell^p_{q_1}$ with $\|\cdot\|_{w\ell^p_{q_1}}\leq\|\cdot\|_{w\ell^p_{q_2}}$.*

2.2 Inclusion relation between discrete Morrey spaces and weak type discrete Morrey spaces

Theorem 2.5. *Let $1\leq p_1\leq q_1<\infty$ and $1\leq p_2\leq q_2<\infty$. If $q_2\leq q_1$ and $p_1<p_2$, then the inclusion $w\ell^{p_2}_{q_2}\subseteq\ell^{p_1}_{q_1}$ is proper.*

Remark 2.6. In view of Theorem 2.5 and the inclusion $\ell^{p_2}_{q_2}\subseteq w\ell^{p_2}_{q_2}$, we see that the inclusion $\ell^{p_2}_{q_2}\subseteq\ell^{p_1}_{q_1}$ is proper if $q_2\leq q_1$ and $p_1<p_2$.

Proof of Theorem 2.5. Suppose that $q_2\leq q_1$ and $p_1<p_2$. Let $x\in w\ell^{p_2}_{q_2}$. We shall show that there exists $C>0$ such that

$$|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\Big(\sum_{j\in S_{m,N}}|x_j|^{p_1}\Big)^{\frac{1}{p_1}}\leq C\|x\|_{w\ell^{p_2}_{q_2}} \tag{2.9}$$

for every $m\in\mathbb{Z}$ and $N\in\mathbb{N}_0$. Observe that

$$\sum_{j\in S_{m,N}}|x_j|^{p_1}=p_1\int_0^\infty t^{p_1-1}|\{j\in S_{m,N}:|x_j|>t\}|\,dt=I_1(R)+I_2(R),$$

where

$$I_1(R):=p_1\int_0^R t^{p_1-1}|\{j\in S_{m,N}:|x_j|>t\}|\,dt,$$
$$I_2(R):=p_1\int_R^\infty t^{p_1-1}|\{j\in S_{m,N}:|x_j|>t\}|\,dt,$$

and $R>0$ is chosen later. The estimate for $I_1(R)$ is

$$I_1(R)\leq p_1\int_0^R t^{p_1-1}|S_{m,N}|\,dt=p_1(2N+1)\int_0^R t^{p_1-1}\,dt=(2N+1)R^{p_1}. \tag{2.10}$$

Meanwhile, by using Definition 2.1 and $p_1<p_2$, we have

$$I_2(R)\leq p_1\int_R^\infty t^{p_1-1}|S_{m,N}|^{1-\frac{p_2}{q_2}}t^{-p_2}\|x\|^{p_2}_{w\ell^{p_2}_{q_2}}\,dt=p_1(2N+1)^{1-\frac{p_2}{q_2}}\|x\|^{p_2}_{w\ell^{p_2}_{q}}\int_R^\infty t^{p_1-p_2-1}\,dt$$
$$=\frac{p_1}{p_2-p_1}(2N+1)^{1-\frac{p_2}{q_2}}\|x\|^{p_2}_{w\ell^{p_2}_{q_2}}R^{p_1-p_2}. \tag{2.11}$$

Combining (2.10) and (2.11) and taking

$$R := \Big(\frac{p_1}{p_2 - p_1}\Big)^{\frac{1}{p_2}} \frac{\|x\|_{w\ell^{p_2}_{q_2}}}{(2N+1)^{\frac{1}{q_2}}},$$

we get

$$\sum_{j \in S_{m,N}} |x_j|^{p_1} \le C(2N+1)^{1-\frac{p_1}{q_2}} \|x\|^{p_1}_{w\ell^{p_2}_{q_2}}. \tag{2.12}$$

Hence, (2.9) follows by taking the p_1-th root of (2.12) and multiplying by $|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}$. Since (2.9) holds for arbitrary $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$, we have $x \in \ell^{p_1}_{q_1}$ with

$$\|x\|_{\ell^{p_1}_{q_1}} \le C\|x\|_{w\ell^{p_2}_{q_2}},$$

as desired. We shall now prove that the inclusion is proper. Note that our assumptions imply $\frac{p_1}{q_1} < \frac{p_2}{q_2}$. Therefore, we can choose $v, w \in \mathbb{N}$ such that

$$\Big(\frac{q_2}{p_2} - 1\Big)w + \frac{2q_2}{p_2} < v < \Big(\frac{q_1}{p_1} - 1\Big)w + 2.$$

Let k_0 be a smallest positive integer such that $1 - \frac{1}{2^{2k_0}} > \frac{1}{2^{v+w-1}}$. Define $x = (x_j)_{j\in\mathbb{Z}}$ by the formula

$$x_j := \begin{cases} 1, & |j| = 0, 1, 2, \dots, 2^{v+w}, \\ 1, & |j| = 2^{k(v+w)}, 2^{k(v+w)} - 2^{kw}, 2^{k(v+w)} - 2(2^{kw}), 2^{k(v+w)} - 3(2^{kw}), \\ & \qquad \dots, 2^{k(v+w)} - (2^{k(v-2)})(2^{kw}), \text{ where } k \in \mathbb{N} \cap [k_0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

By the same argumentation as in the proof of Theorem 2.2, we obtain $x \in \ell^{p_1}_{q_1}$ but $x \notin \ell^{p_2}_{q_2}$. Moreover, repeating the calculation in the proof of Theorem 2.2, we obtain

$$\|x\|_{w\ell^{p_2}_{q_2}} \ge \frac{1}{2}\|x\|_{\ell^{p_2}_{q_2}},$$

so that $x \notin w\ell^{p_2}_{q_2}$. This shows $x \in \ell^{p_1}_{q_1} \setminus w\ell^{p_2}_{q_2}$. Thus, we conclude that the inclusion $w\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$ is proper. □

As a consequence of Theorem 2.2, we have the following necessary condition for inclusion of weak type discrete Morrey spaces into discrete Morrey space.

Theorem 2.7. *Let $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$. Then the inclusion $w\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$ implies $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$.*

Proof. Let $w\ell^{p_2}_{q_2} \subseteq \ell^{p_1}_{q_1}$. Since $\ell^{p_1}_{q_1} \subseteq w\ell^{p_1}_{q_1}$, we have $w\ell^{p_2}_{q_2} \subseteq w\ell^{p_1}_{q_1}$. Therefore, by virtue of Theorem 2.5, we have $q_2 \le q_1$ and $\frac{p_1}{q_1} \le \frac{p_2}{q_2}$. □

Taking $q_1 = q_2 = q$, we obtain the following corollary.

Corollary 2.8. *Let $1 \le p_1 \le q < \infty$ and $1 \le p_2 \le q < \infty$. If the inclusion $w\ell^{p_2}_{q} \subseteq \ell^{p_1}_{q}$ is proper, then $p_1 < p_2$.*

Proof. As a consequence of Theorem 2.7, we have $p_1 \le p_2$. Assume to the contrary that $p_1 = p_2$. Then $\ell^{p_1}_{q} = \ell^{p_2}_{q} \subseteq w\ell^{p_2}_{q}$. This contradicts the fact that $w\ell^{p_2}_{q} \subseteq \ell^{p_1}_{q}$ is proper. Thus, $p_1 < p_2$. □

3 Relation between discrete Morrey spaces and their continuous counterpart

Our results are based on some recent results in [8, 9].

In this section, we discuss the relation between the inclusion property of discrete Morrey spaces and that of Morrey spaces. In particular, we reprove the inclusion property of discrete Morrey spaces by using the inclusion property of Morrey spaces, and we recover some necessary conditions for the inclusion property of Morrey spaces.

We now recall some definitions and notation. Let $1 \le p \le q < \infty$. The Morrey space $\mathcal{M}_q^p(\mathbb{R})$ is defined to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R})$ for which

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}, r>0} (2r)^{\frac{1}{q}-\frac{1}{p}} \Bigg(\int_{a-r}^{a+r} |f(t)|^p \, dt \Bigg)^{\frac{1}{p}}$$

is finite. The weak Morrey space $w\mathcal{M}_q^p(\mathbb{R})$ is defined to be the set of all measurable functions f on $\mathbb{R}$ such that the quasi-norm

$$\|f\|_{w\mathcal{M}_q^p} := \sup_{\lambda>0} \|\lambda \chi_{\{x\in\mathbb{R}\,:\,|f(x)|>\lambda\}}\|_{\mathcal{M}_q^p}$$

is finite. From these definitions, it is clear that $\mathcal{M}_q^p(\mathbb{R}) \subseteq w\mathcal{M}_q^p(\mathbb{R})$. The relation between ℓ_q^p and $\mathcal{M}_q^p(\mathbb{R})$ is given in [8, Remark 2.4] and [9]. For the convenience of the reader, we give the detail here. For every sequence $x = (x_j)_{j\in\mathbb{Z}}$, define a function $\overline{x} : \mathbb{R} \to [0, \infty)$ by

$$\overline{x}(t) := \Big(\sum_{j\in\mathbb{Z}} |x_j|^p \chi_{[j,j+1)}(t) \Big)^{\frac{1}{p}}.$$

Then the discrete Morrey space ℓ_q^p can be realized as a closed subspace of $\mathcal{M}_q^p(\mathbb{R})$ in the following sense.

Theorem 3.1 ([9, Theorem 2.1]). *Let $1 \le p \le q < \infty$. Then there exist positive constants C_1 and C_2 such that*

$$\|\overline{x}\|_{\mathcal{M}_q^p} \le C_1 \|x\|_{\ell_q^p} \tag{3.1}$$

for every $x \in \ell_q^p$ and

$$\|y\|_{\ell_q^p} \le C_2 \|\overline{y}\|_{\mathcal{M}_q^p} \tag{3.2}$$

for every sequence $y = (y_j)_{j\in\mathbb{Z}}$.

An analogous result for weak type discrete Morrey spaces is presented in the following theorem.

Theorem 3.2 ([9, Theorem 2.3]). *Let $1 \le p \le q < \infty$. Then there exist positive constants C_1 and C_2 such that for every $x \in w\ell_q^p$ and for every sequence $y = (y_j)_{j\in\mathbb{Z}}$ the following inequalities hold:*

$$\|\overline{x}\|_{w\mathcal{M}_q^p} \le C_1 \|x\|_{w\ell_q^p} \tag{3.3}$$

and

$$\|y\|_{w\ell_q^p} \le C_2 \|\overline{y}\|_{w\mathcal{M}_q^p}. \tag{3.4}$$

We apply Theorems 3.1 and 3.2 with the inclusion of Morrey spaces to recover the inclusion of discrete Morrey spaces obtained in [5].

Corollary 3.3 ([5]). *Let $1 \le p_1 \le p_2 \le q < \infty$. Then $\ell_q^{p_2} \subseteq \ell_q^{p_1}$ and $w\ell_q^{p_2} \subseteq w\ell_q^{p_1}$.*

Proof. Our proof is an alternative to that in [5]. Let $x \in \ell_q^{p_2}$. Then it follows from (3.1) that $\overline{x} \in \mathcal{M}_q^{p_2}$. Since $p_1 \le p_2$, we have $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$, so that $\overline{x} \in \mathcal{M}_q^{p_1}$. According to (3.2), we have $x \in \ell_q^{p_1}$. Thus, $\ell_q^{p_2} \subseteq \ell_q^{p_1}$. To prove the inclusion $w\ell_q^{p_2} \subseteq w\ell_q^{p_1}$, let $y \in w\ell_q^{p_2}$. Then, as a consequence of (3.3), we have $\overline{y} \in w\mathcal{M}_q^{p_2}$. The inclusion $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$ implies $\overline{y} \in w\mathcal{M}_q^{p_1}$. By virtue of (3.4), we have $y \in w\ell_q^{p_1}$. Thus, $w\ell_q^{p_2} \subseteq w\ell_q^{p_1}$. □

As a consequence of Theorems 1.1, 2.2, 3.1 and 3.2, we may recover a necessary condition for the inclusion property of Morrey spaces and that of weak Morrey spaces obtained in [2].

Corollary 3.4 ([2, Theorem 1.6]). *Let $1 \le p_1 \le q < \infty$ and $1 \le p_2 \le q < \infty$.*
(i) *If $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$, then $p_1 \le p_2$.*
(ii) *If $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$, then $p_1 \le p_2$.*

Proof. (i) Let $x \in \ell_q^{p_2}$. Then, according to Theorem 3.1, we have $\overline{x} \in \mathcal{M}_q^{p_2}$. Since $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$, we obtain $\overline{x} \in \mathcal{M}_q^{p_1}$. This fact and (3.2) imply $x \in \ell_q^{p_1}$. Therefore, $\ell_q^{p_2} \subseteq \ell_q^{p_1}$. Consequently, by virtue of Theorem 1.1, we conclude that $p_1 \le p_2$.

(ii) Let $x \in w\ell_q^{p_2}$. By virtue of Theorem 3.2, we have $\overline{x} \in w\mathcal{M}_q^{p_2}$, so that $\overline{x} \in w\mathcal{M}_q^{p_1}$. Combining this with (3.4), we obtain $x \in w\ell_q^{p_1}$. Consequently, $w\ell_q^{p_2} \subseteq w\ell_q^{p_1}$. Thus, the conclusion follows from Theorem 2.2. □

Our final result is the necessary condition for inclusion between weak Morrey spaces and Morrey spaces that can be viewed as a complement of [4, Theorem 1.1].

Corollary 3.5. *Let* $1 \le p_1 \le q < \infty$ *and* $1 \le p_2 \le q < \infty$. *If* $w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$, *then* $p_1 < p_2$.

Proof. Since $\mathcal{M}_q^{p_1} \subseteq w\mathcal{M}_q^{p_1}$, we have $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$. According to Corollary 3.4, we have $p_1 \le p_2$. If $p_1 = p_2$, then $w\mathcal{M}_q^{p_1} = w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$, so $w\mathcal{M}_q^{p_1} = \mathcal{M}_q^{p_1}$. This contradicts the fact that $\mathcal{M}_q^{p_1}$ is a proper subset of $w\mathcal{M}_q^{p_1}$ (see [4, Theorem 1.2]). Thus, $p_1 < p_2$. □

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