VON NEUMANN CONSTANT FOR WEAK ORLICZ SPACES AND WEAK LEBESGUE SPACES

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Abstract. In this paper we give some estimates for lower bound of von Neumann-Jordan constant for weak Orlicz spaces and weak Lebesgue spaces. As an application, we prove that the von Neumann-Jordan constant for the weak Lebesgue space wL^p tends to 2 as p tends to infinity. Our proof uses the refinement of the positive constant in the triangle inequality in wL^p.

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1. Introduction

Let X be a Banach space. The von Neumann-Jordan constant for X (see [1, 3]) is defined by

$$C_{NJ}(X) := \sup \left\{ \frac{\|f-g\|_X^2 + \|f+g\|_X^2}{2(\|f\|_X^2 + \|g\|_X^2)} : f, g \in X \setminus \{0\} \right\}. \tag{1}$$

It follows from the triangle inequality and also arithmetic and quadratic mean inequality that $C_{NJ}(X) \leq 2$. In addition, by taking $f = g$ in the definition above, we have $C_{NJ}(X) \geq 1$. Moreover, this inequality becomes an equality when X is a Hilbert spaces, In particular, $C_{NJ}(L^2(\mathbb{R}^n)) = 1$. For general $p \in [1, \infty]$, it is known that $C_{NJ}(L^p(\mathbb{R}^n)) = \text{maks}\left\{2^{\frac{2}{p}-1}, 2^{1-\frac{2}{p}}\right\}$ for finite p and $C_{NJ}(L^\infty(\mathbb{R}^n)) = 2$.

The study of von Neumann-Jordan of Lebesgue spaces can be generalized to Orlicz spaces. Let us recall the definition of these spaces (see [5]). Let $\Phi : [0, \infty) \to [0, \infty)$ be any N-function, namely Φ is convex, $\Phi(0) = 0$, $\lim\limits_{t\to 0} \frac{\Phi(t)}{t} = 0$, and $\lim\limits_{t\to\infty} \frac{\phi(t)}{t} = \infty$. The Orlicz space $L^\Phi(\mathbb{R}^n)$ is defined to be the set of all measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{L^\Phi} := \inf \left\{ \lambda > 0 : \int_{R^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \right\} < \infty.$$

For $\Phi(t) = t^p$, where $1 \leq p < \infty$, we have $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Some results on von Neumann-Jordan constant of Orlicz spaces can be seen in [7]. One of the results in this book is

$$C_{NJ}(L^\Phi(\mathbb{R}^n)) \geq \max\left\{ \frac{1}{\bar{\alpha}_\Phi}, 2\bar{\beta}_\Phi^2 \right\},$$

where $\bar{\alpha}_\Phi$ and $\bar{\beta}_\Phi$ are defined by $\bar{\alpha}_\Phi := \inf_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(2t)}$ and $\bar{\beta}_\Phi = \sup_{t>0} \frac{\Phi^{-1}(t)}{\Phi^{-1}(2t)}$. In this paper, we investigate the von Neumann-Jordan constant for weak Orlicz spaces and weak Lebesgue spaces. Recall that The weak Orlicz space $wL^\Phi(\mathbb{R}^n)$ is defined to be the set of measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{wL^\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \sup_{t>0} \Phi(t) \left| \left\{ x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t \right\} \right| \leq 1 \right\} < \infty.$$

Note that, $L^\Phi(\mathbb{R}^n) \subseteq wL^\Phi(\mathbb{R}^n)$ and $\|f\|_{wL^\Phi} = \sup_{t>0} \|t\chi_{\{|f|>t\}}\|_{L^\Phi}$. Meanwhile, the weak Lebesgue space $wL^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is defined to be the collection of measurable functions f on $\mathbb{R}^n$ for which

$$||f||_{wL^p} = \sup_{\gamma>0} \gamma(\{x \in \mathbb{R}^n \ : \ |f(x)| > \gamma\})^{\frac{1}{p}} < \infty.$$

If $\Phi(t) := t^p$ for some $1 \leq p < \infty$, then $wL_\Phi(\mathbb{R}^n) = wL^p(\mathbb{R}^n)$. Thus, $L_\Phi(\mathbb{R}^n)$ can be view as a generalization of the weak Lebesgue space $L^p(\mathbb{R}^n)$. Note that

$$\|f\|_{wL_\Phi(\mathbb{R}^n)} := \sup_{t>0} \|t \ \chi_{\{|f|>t\}}\|_{L_\Phi(\mathbb{R}^n)}.$$

For a quasi-Banach space X, the von Neumann-Jordan constant $C_{NJ}(X)$ is defined by

$$C_{NJ}(X) := \sup \left\{ \frac{||f+g||_X^2 + ||f-g||_X^2}{2C_X^2(||f||_X^2 + ||g||_X^2)} \ : \ f, g \in X, \ \text{not both zero} \right\} \tag{2}$$

where

$$C_X = \sup \left\{ \frac{||f+g||_X}{||f||_X + ||g||_X} : f, g \in X, (f,g) \neq (0,0) \right\}.$$

See [6] for some related results. Observe that, if X is a Banach space, then $C_X = 1$, so that we can recover (1) from (2). One of our main results is the following lower bound of von Neumann-Jordan constant of weak Orlicz spaces.

Theorem 1.1. *Let Φ be any N-function and write $C_\Phi := C_{wL^\Phi(\mathbb{R}^n)}$. Then*

$$\max \left(\frac{1}{2C_\Phi^2 \bar{\alpha}_\Phi^2}, \frac{2\bar{\beta}_\Phi^2}{C_\Phi^2} \right) \leq C_{NJ}(wL^\Phi(\mathbb{R}^n)) \tag{3}$$

As a consequence of Theorem 1.1, we obtain a lower bound for von Neumann-Jordan constant of weak Lebesgue spaces and the asymptotic value of these constants as $p \to \infty$.

Theorem 1.2. *If $1 \leq p < \infty$, then*

$$\max \left(\frac{2^{\frac{2}{p}} - 1}{C_p^2}, \frac{2^{1-\frac{2}{p}}}{C_p^2} \right) \leq C_{NJ}(wL^p(\mathbb{R}^n)) \leq 2, \tag{4}$$

where $2^{\frac{1}{p}} \leq C_p \leq \min \left\{ 2, \frac{p}{p-1} \right\}$. In particular, $\lim_{p\to\infty} C_{NJ}(wL^p(\mathbb{R}^n)) = 2$.

2. Preliminaries

Note that, the weak Orlicz space $wL^{\Phi}(\mathbb{R}^n)$ always contains the characteristic function on set of finite measure.

Lemma 2.1. [4] *Let E be a measurable set of $\mathbb{R}^n$ and $0 < |E| < \infty$. Then*

$$||\chi_E||_{wL^{\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{|E|})}$$

Lemma 2.2. *Let $1 \le p < \infty$ and define*

$$C_p = \sup_{(f,g)\neq(0,0)} \frac{||f+g||_{wL^p}}{||f||_{wL^p} + ||g||_{wL^p}}.$$

Then

$$2^{\frac{1}{p}} \le C_p \le 2.$$

Proof. Observe that, for every $t > 0$, we have

$$\begin{aligned}|\{x \in \mathbb{R}^n : |f(x) + g(x)| > t\}| &\le |\{x \in \mathbb{R}^n : |f(x)| > \frac{t}{2}\}| + |\{x \in \mathbb{R}^n : |g(x)| > t\}| \\ &= \left(\frac{t}{2}\right)^{-p} ||f||^p_{wL^p} + \left(\frac{t}{2}\right)^{-p} ||g||^p_{wL^p} \\ &= t^{-p}2^p(||f||^p_{wL^p} + ||g||^p_{wL^p})\end{aligned}$$

Multiplying by t^p and taking the p-th root, we obtain

$$\begin{aligned}t|\{x \in \mathbb{R}^n : |f(x) + g(x)| > t\}|^{1/p} &\le 2(||f||^p_{wL^p} + ||g||^p_{wL^p})^{1/p} \\ &\le 2(||f||_{wL^p} + ||g||_{wL^p})\end{aligned}$$

By taking the supremum over $t > 0$, we get

$$||f + g||_{wL^p} \le 2(||f||_{wL^p} + ||g||_{wL^p}) \tag{5}$$

Let $f(x) = x^{-\frac{1}{p}}\chi_{(0,1)}$ and $g(x) = (1-x)^{-\frac{1}{p}}\chi_{(0,1)}$. Then

$$||f||_{wL_p} = ||g||_{wL^p} = 1.$$

Let $h(x) = f(x) + g(x)$. Therefore we get,

$$\begin{aligned}h(a)|\{x : |h(x) > h(a)\}|^{\frac{1}{p}} &= \left(a^{-\frac{1}{p}} + (1-a)^{-\frac{1}{p}}\right)(2a)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}}\left(1 + (\frac{a}{1-a})^{\frac{1}{p}}\right).\end{aligned}$$

By taking the supremum over a, we have $||f + g||_{wL^p} = 2^{\frac{1}{p}}.2 = 2^{1+\frac{1}{p}}$. This implies

$$C_p \ge \frac{||f+g||_{wL^p}}{||f||_{wL^p} + ||g||_{wL^p}} = \frac{2^{1+\frac{1}{p}}}{2} = 2^{\frac{1}{p}}. \tag{6}$$

Combining (5) and (6), we get $2^{\frac{1}{p}} \le C_p \le 2$. □

Proposition 2.3. [2] *Let $1 < p < \infty$ and define*

$$||f||^*_{wL^p} = \sup_{|E|\subset\mathbb{R}^n} |E|^{\frac{1}{p}-1}\left(\int_{R^n} |f(x)|\, dx\right).$$

Then we have

$$||f||^*_{wL^p} \leq \frac{p}{p-1}||f||_{wL^p}.$$

Proof. Let $0 < |E| < \infty$.

$$\begin{aligned}\int_E |f(x)|\ dx &= \int_E \int_0^{|f(x)|}\ dt\ dx \\ &= \int_0^\infty \int_{\{x\in E:|f(x)|>t\}}\ dx\ dt \\ &= \int_0^R \{x \in E : |f(x)| > t\}\ dt + \int_R^\infty \{x \in E : |f(x)| > t\}\ dt \\ &\leq \int_0^R |E|\ dt + \int_R^\infty t^{-p}||f||^p_{wL^p}\ dt \\ &= R|E| + \frac{1}{p-1}R^{-p+1}||f||^p_{wL^p}\end{aligned}$$

Let $g(R) = R|E| + \frac{1}{p-1}R^{-p+1}||f||^p_{wL^p}$. We have $g'(R) = |E| + \frac{1-p}{p-1}R^{-p}||f||^p_{wL^p}$. So $g'(R) = 0 \Leftrightarrow R^{-p}||f||^p_{wL^p} = |E|$ or $R = \frac{||f||^p_{wL^p}}{|E|^{1/p}}$.

$$\begin{aligned}g(R) &= \frac{||f||^p_{wL^p}}{|E|^{1/p}}|E| + \frac{1}{p-1}\left(\frac{||f||^p_{wL^p}}{|E|^{1/p}}\right)^{1-p}||f||_{wL^p} \\ &= ||f||_{wL^p}|E|^{1-\frac{1}{p}} + \frac{1}{p-1}||f||_{wL^p}|E|^{1-\frac{1}{p}} \\ &= \Big(1 + \frac{1}{p-1}||f||_{wL^p}|E|^{1-\frac{1}{p}} \\ &= \frac{p}{p-1}||f||_{wL^p}|E|^{1-\frac{1}{p}}.\end{aligned}$$

Therefore, we get

$$|E|^{\frac{1}{p}-1}\int_E |f(x)|\ dx \leq \frac{p}{p-1}||f||_{wL^p}.$$

By taking the supremum over E we conclude that

$$||f||^*_{wL^p} \leq \frac{p}{p-1}||f||_{wL^p}.$$

□

Proposition 2.4. [2] *Let $p > 1$. Then we have*

$$||f||_{wL^p} \leq ||f||^*_{wL^p}.$$

Proof. Define $E = \{|f(x)| > t\}$. Note that for every $t > 0$

$$|E| \leq t^{-p}||f||_{wL^p} < \infty.$$

Therefore,

$$|E|^{\frac{1}{p}-1}\int_E |f(x)|\ dx \leq ||f||^*_{wL^p}. \tag{7}$$

On the other hand

$$|E|^{\frac{1}{p}-1}\int_E |f(x)|\,dx = |\{x\in\mathbb{R}^n : |f(x)|>t\}|^{\frac{1}{p}-1}\left(\int_{\{x:|f(x)|>t\}} |f(x)|\,dx\right)$$
$$= |\{x\in\mathbb{R}^n : |f(x)|>t\}|^{\frac{1}{p}-1} t|\{x : |f(x)|>t\}|$$
$$\geq t|\{x\in\mathbb{R}^n : |f(x)|>t\}|^{\frac{1}{p}} \tag{8}$$

Combining (7) and (8), we get

$$t|\{x\in\mathbb{R}^n : |f(x)|>t\}|^{\frac{1}{p}} \leq ||f||^*_{wL^p}.$$

Since $t>0$ is arbitrary, we conclude that

$$||f||_{wL^p} \leq ||f||^*_{wL^p}. \tag{9}$$

□

Proposition 2.5. *Let $p>1$. Define*

$$C_p = \sup_{(f,g)\neq(0,0)} \frac{||f+g||_{wL^p}}{||f||_{wL^p}+||g||_{wL^p}}.$$

Then

$$2^{\frac{1}{p}} \leq C_p \leq \min\left\{2, \frac{p}{p-1}\right\}. \tag{10}$$

Proof. From (9) and normability of weak L^p we have

$$||f+g||_{wL^p} \leq ||f+g||^*_{wL^p}$$
$$\leq ||f||^*_{wL^p} + ||g||^*_{wL^p}$$
$$\leq \left(\frac{p}{p-1}\right)||f||_{wL^p} + \left(\frac{p}{p-1}\right)||g||_{wL^p}$$
$$= \left(\frac{p}{p-1}\right)\left(||f||_{wL^p}+||g||_{wL^p}\right). \tag{11}$$

Combining (5) and (11), we conclude that

$$C_p = \sup_{(f,g)\neq(0,0)} \frac{||f+g||_{wL^p}}{||f||_{wL^p}+||g||_{wL^p}} \leq \min\left\{2, \frac{p}{p-1}\right\}. \tag{12}$$

By (6) and (12), we conclude that $2^{\frac{1}{p}} \leq C_p \leq \min\left\{2, \frac{p}{p-1}\right\}$. □

3. Proofs of Main Results

We first prove Theorem 1.1.

Proof of Theorem 1.1. According to the definition of $\bar{\alpha}_\Phi$, for any $\varepsilon>0$, there exists $t_0>0$ such that

$$\frac{\Phi^{-1}(t_0)}{\Phi^{-1}(2t_0)} < \bar{\alpha}_\Phi + \epsilon. \tag{13}$$

Define $r_0 := (2t_0|B(0,1))^{-1/n}$ and choose $x_1, x_2 \in \mathbb{R}^n$ such that $B(x_1, r_0)$ and $B(x_1, r_0)$ are disjoint. For $i = 1, 2$, set $f_i := \Phi^{-1}(2t_0)\chi_{B(x_i,r_0)}$. Then, by Lemma 2.1, we have

$$\|f_i\|_{wL^\phi} = \Phi^{-1}(2t_0)\|B(x_i, r_0)\|_{wL^\Phi} = \frac{\Phi^{-1}(2t_0)}{\Phi^{-1}(1/|B(x_i, r_0)|)} = 1. \tag{14}$$

Since $B(x_1, r_0)$ and $B(x_1, r_0)$ are disjoint, we have

$$\begin{aligned} ||f_1 + f_2||_{wL^\Phi} &= \Phi^{-1}(2t_0)\|\chi_{B(x_1,r_0)} + \chi_{B(x_2,r_0)}\|_{wL^\Phi} \\ &= \Phi^{-1}(2t_0)\|\chi_{B(x_1,r_0)\cup B(x_2,r_0)}\|_{wL^\Phi} \\ &= \frac{\Phi^{-1}(2t_0)}{\Phi^{-1}\left(\frac{1}{2|B(x_1,r_0)|}\right)} = \frac{\Phi^{-1}(2t_0)}{\Phi^{-1}(t_0)} > \frac{1}{\bar{\alpha}_\Phi + \varepsilon}. \end{aligned} \tag{15}$$

Similarly,

$$||f_1 - f_2||_{wL^\Phi} > \frac{1}{\bar{\alpha}_\Phi + \varepsilon}. \tag{16}$$

Combining (14)–(16), we get

$$C_{NJ}(wL^\Phi(\mathbb{R}^n)) \geq \frac{||f_1 + f_2||^2_{wL^\Phi} + ||f_1 - f_2||^2_{wL^\Phi}}{2C^2_\Phi(||f_1||^2_{wL^\Phi} + ||f_2||^2_{wL^\Phi})} \geq \frac{1}{2C^2_\Phi(\bar{\alpha}_\Phi + \epsilon)^2}$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$C_{NJ}(wL^\Phi(\mathbb{R}^n)) \geq \frac{1}{2C^2_\Phi \bar{\alpha}^2_\Phi}. \tag{17}$$

Similarly, by definition of $\bar{\beta}_\Phi$, for any $\varepsilon > 0$, there exists $u_0 > 0$ such that

$$\frac{\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} > \bar{\beta}_\Phi - \frac{\varepsilon}{2} \tag{18}$$

Set $v_0 := (2u_0|B(0,1)|)^{-1/n}$. Define $g_1 := \Phi^{-1}(u_0)(\chi_{B(y_1,v_0)} + \chi_{B(y_2,v_0)})$ and $g_2 := \Phi^{-1}(u_0)(\chi_{B(y_1,v_0)} - \chi_{B(y_2,v_0)})$, where $B(y_1, v_0)$ and $B(y_2, v_0)$ are disjoint.

Observe that, by Lemma 2.1, we have

$$\begin{aligned} \|g_1\|_{wL^\phi} &= \Phi^{-1}(u_0)\|\chi_{B(y_1,v_0)\cup B(y_2,v_0)}\|_{wL^\Phi} \\ &= \frac{\Phi^{-1}(u_0)}{\Phi^{-1}\left(\frac{1}{|B(y_1,v_0)\cup B(y_2,v_0)|}\right)} = \frac{\Phi^{-1}(u_0)}{\Phi^{-1}\left(\frac{1}{2|B(y_1,v_0)|}\right)} = 1. \end{aligned} \tag{19}$$

By a similar argument, we have $\|g_2\|_{wL^\Phi} = 1$,

$$\|g_1 + g_2\|_{wL^\Phi} = 2\Phi^{-1}(u_0)\|\chi_{B(y_1,v_0)}\|_{wL^\Phi} = \frac{2\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} > 2\bar{\beta} - \varepsilon$$

and

$$\|g_1 - g_2\|_{wL^\Phi} > 2\bar{\beta} - \varepsilon$$

$$\|f_i\|_{wL^\Phi} = \Phi^{-1}(u_0)\|\chi_{B(y_i,v_0)}\|_{wL^\Phi} = \frac{\Phi^{-1}(u_0)}{\Phi^{-1}(|B(y_i, v_0)|^{-1})} = 1.$$

These estimates yield

$$C_{NJ}(wL^{\Phi}(\mathbb{R}^n)) \geq \frac{||g_1+g_2||^2_{wL^{\Phi}} + ||g_1-g_2||^2_{wL^{\Phi}}}{2C_{\Phi}^2(||g_1||^2_{wL^{\Phi}} + ||g_2||^2_{wL^{\Phi}})} > \frac{(2\bar{\beta}_{\Phi}-\varepsilon)^2}{C_{\Phi}^2}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$C_{NJ}(wL^{\Phi}(\mathbb{R}^n)) \geq \frac{2\bar{\beta}_{\Phi}^2}{C_{\Phi}^2} \tag{20}$$

Thus, (3) follows from (17) and (20). □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We first prove the lower bound in the inequality (4). Let $\phi(u) = u^p$. So that

$$\bar{\alpha}_{\Phi} = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \inf_{u>0} \frac{u^{\frac{1}{p}}}{(2u)^{\frac{1}{p}}} = 2^{-\frac{1}{p}}$$

and

$$\bar{\beta}_{\Phi} = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = \sup_{u>0} \frac{u^{\frac{1}{p}}}{(2u)^{\frac{1}{p}}} = 2^{-\frac{1}{p}}.$$

For $\phi(u) = u^p$, we have $wL^{\Phi}(\mathbb{R}^n) = wL^p(\mathbb{R}^n)$. By (3) we get

$$C_{NJ}(wL^p(\mathbb{R}^n)) \geq \max\left(\frac{2^{\frac{2}{p}}-1}{C_p^2}, \frac{2^{1-\frac{2}{p}}}{C_p^2}\right). \tag{21}$$

Note that

$$\begin{aligned} ||f+g||^2_{wL^p} + ||f-g||^2_{wL^p} &\leq 2(C_p[||f||^2_{wL^p} + ||g||^2_{wL^p}])^2 \\ &= 2C_p^2(||f||^2_{wL^p} + ||g||^2_{wL^p})^2 \\ &\leq 4C_p^2(||f||^2_{wL^p} + ||g||^2_{wL^p}), \end{aligned}$$

Therefore, by definition of $C_{NJ}(wL^p(\mathbb{R}^n))$ we get

$$C_{NJ}(wL^p(\mathbb{R}^n)) \leq 2. \tag{22}$$

We combine the inequalities (21) and (22) to obtain

$$\max\left(\frac{2^{\frac{2}{p}}-1}{C_p^2}, \frac{2^{1-\frac{2}{p}}}{C_p^2}\right) \leq C_{NJ}(wL^p) \leq 2.$$

We now prove the second part of Theorem 1.2.

From (10), we have

$$\begin{gathered} C_p = 1, p = 1 \\ \sqrt{2} \leq C_p \leq 2, p \in (1,2] \\ 2^{\frac{1}{p}} \leq C_p \leq \frac{p}{p-1}, p > 2. \end{gathered}$$

Therefore, by (21), for $p = 1$ we get

$$C_{NJ}((\mathbb{R}^n)) \geq \max\left\{\frac{1}{2}, \frac{1}{8}\right\} = \frac{1}{2}.$$

For $1 < p \leq 2$ we have

$$C_{NJ}(wL^p(\mathbb{R}^n)) \geq \frac{2^{\frac{2}{p}-1}}{C_p^2} \in \left[2^{\frac{2}{p}-3}, \frac{1}{2}\right]$$

and for $p > 2$ we have

$$C_{NJ}((\mathbb{R}^n)) \geq \frac{2^{1-\frac{2}{p}}}{C_p^2} \in \left[\frac{2^{1-\frac{2}{p}}}{\left(\frac{p}{p-1}\right)^2}, 2^{1-\frac{4}{p}}\right] \tag{23}$$

Finally, for large p, we get

$$\lim_{p\to\infty} C_{NJ}(wL^p(\mathbb{R}^n)) = 2.$$

□

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