Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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To cite this article: H Batkunde *et al* 2018 *J. Phys.: Conf. Ser.* **1097** 012079

IOP Conf. Series: Journal of Physics: Conf. Series **1097** (2018) 012079 doi:10.1088/1742-6596/1097/1/012079

n-Normed Spaces with Norms of Its Quotient Spaces

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Abstract. The concept of n-normed spaces is a generalization of the concept of normed spaces. Some characteristics of n-normed spaces have been discussed by many researchers. These spaces are usually observed using a set of n linear independent vectors. In this paper, we will construct quotient spaces of an n-normed space with respect to n linear independent vectors. We define a norm in each quotient space by using the n-norm that we have. Norms of these quotient spaces will be a new viewpoint in observing characteristics of n-normed spaces.

1. Introduction

S. Gähler introduced the concept of n-normed spaces for $n \geq 2$ in 1960's [5, 6, 7, 8]. Since then, the structures of these spaces have been studied by many researchers. Most of them observed some characteristics of the n-normed spaces using a set of n linear independent vectors, see for instance [1, 2, 3, 10, 11, 12].

Definition 1.1. [10] Let n be a nonnegative integer and X is a real vector space with $\dim(X) \geq n$. An ***n*-norm** is a function $\|\cdot,\dots,\cdot\|: X^n \to \mathbb{R}$ which satisfies these following conditions:

i. $\|x_1,\dots,x_n\| \geq 0$,
   $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ linearly dependent.
ii. $\|x_1,\dots,x_n\|$ is invariant under permutation.
iii. $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$ for any $\alpha \in \mathbb{R}$.
iv. $\|x_1 + x_1',\dots,x_n\| \leq \|x_1,\dots,x_n\| + \|x_1',\dots,x_n\|$.

The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an ***n*-normed space**.

For example, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, we can define the standard n-norm on X by

$$\|x_1,\dots,x_n\|^S := \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \dots & \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}}.$$

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The above determinant is called Gram's determinant. The value of this determinant will be nonnegative. Geometrically, the value of $\|x_1, \dots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ [9].

On the other hand, if we have a vector space, we can form some equivalence classes with respect to an equivalence relation that has been defined. From these equivalence classes, we can construct several quotient spaces in the given vector space. Whenever possible, we can define a norm on each quotient space.

Let ~ be an equivalence relation on X. For $x \in X$, the set of all elements equivalent to a is denoted by

$$\bar{x} = \{b \in X : \ b \sim x\},$$

and called the **equivalence class** of a.

Let X be a vector space over F, $x \in X$, and $V \subseteq X$. The set $\bar{x} = x + V = \{x + v : v \in V\}$ is called **a coset** of V in X and x is called **a coset representative** for $x + V$. Moreover, the set of all cosets of V in X is denoted by

$$X/V = \{\bar{x} \ : \ x \in X\}.$$

This set is called **the quotient space of *X* modulo V**. [13]

We define an addition and a scalar multiplication operation on X/V, those are $\bar{u} + \bar{w} = \overline{u + w}$ and $\alpha\bar{u} = \overline{\alpha u}$ for $u, w \in X$ and $\alpha \in F$.

Moreover, we will construct quotient spaces of the n-normed spaces with respect to a set of n linear independent vectors. We will also define a norm on each quotient space. These norms will be a new viewpoint in observing the n-normed spaces.

2. Main results

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $Y = \{y_1, \dots, y_n\}$ is a linearly independent set in X. For a fixed $j \in \{1, \dots, n\}$, we consider $Y \backslash \{y_j\}$ and then we define a subspace of X generated by $Y \backslash \{y_j\}$

$$Y_j^0 \coloneqq \text{span } Y \backslash \{y_j\} = \left\{ \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \, y_i \ \ ; \ \ \alpha_i \in \mathbb{R} \right\}.$$

For any $u \in X$, the corresponding coset of Y_j^0 in X is

$$\bar{u} = \left\{ u + \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \, y_i \ \ ; \ \ \alpha_i \in \mathbb{R} \right\}.$$

Hence we have $\bar{0} = \text{span } Y \backslash \{y_j\} = Y_j^0$, and If $\bar{u} = \bar{v}$ then $u - v \in Y_j^0$. Next, define the quotient space of X as

$$X_j^* = X/Y_j^0 = \{\bar{u} : u \in X\}.$$

IOP Conf. Series: Journal of Physics: Conf. Series **1097** (2018) 012079 doi:10.1088/1742-6596/1097/1/012079

Then the addition and the scalar multiplication also apply on X_j^*.

Furthermore we define a function $\|\cdot\|_j^* : X_j^* \to \mathbb{R}$ defined by

$$\|\bar{u}\|_j^* = \|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\|. \tag{1}$$

This function is well defined because for any $\bar{u}, \bar{v} \in X_j^*$ with $\bar{u} = \bar{v}$ we have

$$u = v + \sum_{\substack{i=1 \\ i \neq j}}^{n} \beta_i \, y_i \quad ; \quad \beta_i \in \mathbb{R},$$

so that

$$\|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\| = \left\| v + \sum_{\substack{i=1 \\ i \neq j}}^{n} \beta_i \, y_i \, , y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n \right\|.$$

Therefore

$$\|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\| = \|v, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\|,$$

or

$$\|\bar{u}\|_j^* = \|\bar{v}\|_j^*.$$

This function defines a norm in X_j^*, so that $(X_j^*, \|\cdot\|_j^*)$ is a normed space, as stated in this following theorem.

Theorem 2.1. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space and $Y = \{y_1, \dots, y_n\}$ is a linearly independent set in X. Let $j \in \{1, \dots, n\}$, then $(X_j^*, \|\cdot\|_j^*)$ is a normed space, where $\|\cdot\|_j^*$ is a function defined in (1).

Proof.

We just need to prove that the function we defined in (1) is a norm on X_j^*. Based on the definition that

$$\|\bar{u}\|_j^* = \|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\|,$$

then using the properties of n-norm, one can see that $\|\bar{u}\|_j^* \geq 0$. The function in (1) also satisfies $\|\alpha \, \bar{u}\|_j^* = |\alpha| \| \, \bar{u}\|_j^*$ and triangle inequality $\|\bar{u} + \bar{v}\|_j^* \leq \|\bar{u}\|_j^* + \|\bar{v}\|_j^*$.

Moreover, if $\|\bar{u}\|_j^* = 0$, then $\|u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n\| = 0$. Therefore vectors $u, y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n$ $a$re linearly dependent. Because $Y = \{y_1, \dots, y_n\}$ is a linearly independent set, it follows that

$$u = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \, y_i \, ,$$

with $\alpha_i \in \mathbb{R}$ and $i = 1, \dots, j-1, j+1 \dots, n$. This means that $u \in \text{span } \mathrm{Y} \backslash \{\mathrm{y_j}\}$, so that $\bar{u} = \bar{0}$. Conversely, if $\bar{u} = \bar{0}$, then it is obvious that $\|\bar{u}\|_j^* = 0$. □

By using the above construction, we can get n quotient spaces of a normed space. The idea is to 'eliminate' one vector from Y to get sets $Y \backslash \{y_j\}$ for any $j = 1, \dots, n$. Then we construct the

IOP Conf. Series: Journal of Physics: Conf. Series **1097** (2018) 012079 doi:10.1088/1742-6596/1097/1/012079

corresponding quotient space for each set $Y\backslash\{y_j\}$. These quotient spaces have the same structures but different elements. Moreover, we collect these quotient spaces in a set and name it a **class-1 collection**.

Furthermore, we generalize the above construction by observing $Y\backslash\{y_{i_1},\dots,y_{i_m}\}$. We do so by fixing $i_1,\dots,i_m \in \{1,\dots,n\}$ first. Define a subspace of X generated by $Y\backslash\{y_{i_1},\dots,y_{i_m}\}$,

$$Y^0_{i_1,\dots,i_m} := \text{span } Y\backslash\{y_{i_1},\dots,y_{i_m}\} = \left\{ \sum_{\substack{i=1 \\ i\neq i_1,\dots,i_m}}^{n} \alpha_i\, y_i \ : \alpha_i \in \mathbb{R} \right\}.$$

For any $u \in X$, the corresponding coset in X is

$$\bar{u} = \left\{ u + \sum_{\substack{i=1 \\ i\neq i_1,\dots,i_m}}^{n} \alpha_i\, y_i \ : \alpha_i \in \mathbb{R} \right\}.$$

We have $\bar{0} = \text{ span } Y\backslash\{y_{i_1},\dots,y_{i_m}\} = Y^0_{i_1,\dots,i_m}$ and If $\bar{u} = \bar{v}$, then $u - v \in Y^0_{i_1,\dots,i_m}$. We define quotient space of X as

$$X^*_{i_1,\dots,i_m} = X/Y^0_{i_1,\dots,i_m} := \{\bar{u} : u \in X\}. \tag{2}$$

The addition and the scalar multiplication also apply in this space.

Moreover, define a function $\|\cdot\|^*_{i_1,\dots,i_m}: X^*_{i_1,\dots,i_m} \to \mathbb{R}$ defined by

$$\|\bar{u}\|^*_{i_1,\dots,i_m} = \|u, y_1, \dots, y_{i_1-1}, y_{i_1+1}, \dots, y_n\| + \cdots + \|u, y_1, \dots, y_{i_m-1}, y_{i_m+1}, \dots, y_n\|. \tag{3}$$

The right hand of equation (3) is actually a summation of norms defined in (1). Then we can write equation (3) as

$$\|\bar{u}\|^*_{i_1,\dots,i_m} = \|\bar{u}\|^*_{i_1} + \cdots + \|\bar{u}\|^*_{i_m}. \tag{4}$$

One can see that this function is well defined. This function defines a norm in $X^*_{i_1,\dots,i_m}$, as stated in this following theorem.

Theorem 2.2. Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space and $Y = \{y_1,\dots,y_n\}$ is a linearly independent set in X. Let $1 \le m \le n$, then $\left(X^*_{i_1,\dots,i_m}, \|\bar{u}\|^*_{i_1,\dots,i_m}\right)$ is a normed space, where $\|\cdot\|^*_{i_1,\dots,i_m}$ is a function defined in (3).

Proof.

This proof is analogous with proof of theorem 2.1.

Note that using the above construction, we can get $\binom{n}{m}$ quotient spaces. This time we 'eliminate' any $n - m$ vectors from Y to get $\binom{n}{m}$ sets in the form of $Y\backslash\{y_{i_1},\dots,y_{i_m}\}$ and then we construct the coresponding quotient spaces for each set. These quotient spaces have the same structures but different elements. We collect these quotient spaces in a set and name it a **class-m collection**.

In this discussion, it is very important to identify the vector "0" in the n-normed spaces as a vector not as a coset. This can be done by observing the coset $\overline{0}$ in each quotient space in a class-*m* collection, with $m \in \mathbb{N}, 1 \leq m \leq n$. In other words, we observe the intersection of coset $\overline{0}$ of all quotient spaces in a class-m collection. We apply this to all vectors in the n-normed spaces, so each vector in the n-normed spaces can be identified as a vector not as a coset.

Moreover, we can see some relations between the norms of class-1 collection and class-*m* collection for any $m \in \mathbb{N}, 1 \leq m \leq n$. We show these relations in this following example.

Example 2.3.

a. We observe $(\mathbb{R}^d, \|\cdot,\cdot,\cdot\|)$ as a 3-normed space $(d \geq 3)$ and consider all quotient spaces of class-2 collection here. The norms that we have are

$$\|\cdot\|^*{}_{2,3} = \|\cdot\|_2^* + \|\cdot\|_3^*,$$
$$\|\cdot\|^*{}_{1,3} = \|\cdot\|_1^* + \|\cdot\|_3^*,$$
$$\|\cdot\|^*{}_{1,2} = \|\cdot\|_1^* + \|\cdot\|_2^*.$$

At the same time if we consider the class-1 collection, then for those norms at the right hand of the three equations above they are nothing but norms of class-1 collection. One can notice that from the above equations we can obtain

$$\|\cdot\|^*{}_{2,3} + \|\cdot\|^*{}_{1,3} + \|\cdot\|^*{}_{1,2} = 2\ (\|\cdot\|_1^* + \|\cdot\|_2^* + \|\cdot\|_3^*).$$

Then the summation of all norms of class-2 collection can be written as a summation of norms of class-1 collection.

b. On the other hand, if we observe a 4-normed space and consider the norms in class-3 collection and class-1 collection we can obtain

$$\|\cdot\|^*{}_{2,3,4} + \|\cdot\|^*{}_{1,3,4} + \|\cdot\|^*{}_{1,2,4} + \|\cdot\|^*{}_{1,2,3} = 3\ (\|\cdot\|_1^* + \|\cdot\|_2^* + \|\cdot\|_3^* + \|\cdot\|_4^*).$$

Example 2.3 indicates that there is a relation between class-1 collection and class-*m* collection.

Note that the summation of all norms of class-1 collection is actually the norm of class-*n* collection.

From example 2.3 we can obtain a general relation between the norms of class-1 collection and class-*m* collection for $m \in \mathbb{N}, 1 \leq m \leq n$. We state it in this following theorem.

Theorem 2.4. Suppose that for an n-normed spaces we define the class-*m* collection $(m \in \mathbb{N}, 1 \leq m \leq n)$ in it. Then the relation between norms of class-1 collection and class-*m* collection is

$$\sum \|\cdot\|_{i_1,\dots,i_m} = \binom{n-1}{m-1} (\|\cdot\|_1^* + \cdots + \|\cdot\|_n^*). \qquad (5)$$

The sum is taken over $i_1, \dots, i_m \in \{1, \dots, n\}$ with $i_1 < \cdots < i_m$.

Proof.
The proof is simple, we just need to examine the coefficient of the right hand of equation (5). Based on the fact that

$$\|\cdot\|_{i_1,\dots,i_m} = \|\cdot\|_{i_1}^* + \cdots + \|\cdot\|_{i_m}^*,$$

then the coefficient can be obtained by choosing one norm of class-1 collection and counting how many times the norm appears as a term of the summation in each norm of the class-*m* collection that we observed. This is a combinatorial problem and its solution is $\binom{n-1}{m-1}$. □

Previous researchers who have work on *n*-normed spaces also used summation of norms of class-1 collections to observe characteristics of *n*-normed spaces (see [1, 2, 3, 4, 5]). We can make this simpler using the viewpoint of the theory above.

Consider $(\mathbb{R}^4, \|\cdot,\cdot,\cdot,\cdot\|)$ as a 4-normed spaces, Instead of using 4 norms of class-1 collections, we just use 2-norms of class-2 collections, namely $\|\cdot\|_{1,3}^*$ and $\|\cdot\|_{2,4}^*$. Then we have fewer norms to observe characteristics of *n*-normed spaces.

Generally, we can observe the characteristics of *n*-normed spaces using some norms $\|\cdot\|_{i_1,\dots,i_m}^*$ of class-*m* collection. We just need to choose norms $\|\cdot\|_{i_1,\dots,i_m}^*$ of class-*m* collection such that

$$\bigcup \{i_1, \dots, i_m\} \supseteq \{1, \dots, n\}.$$

This means that for $m \in \mathbb{N}, 1 < m < n$, we do not need to choose all norms of class-*m* collection to observe the *n*-normed spaces.

3. Acknowledgments

This research is supported by ITB research and innovation program 2018.

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