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FEFFERMAN'S INEQUALITY AND UNIQUE CONTINUATION PROPERTY OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we prove a Fefferman's inequality for potentials belonging to a generalized Morrey space $L^{p,\varphi}$ and a Stummel class $\tilde{S}_{\alpha,p}$. Our result extends the previous Fefferman's inequality that was obtained in [3, 7] for the case of Morrey spaces, and that in [22] for the case of Stummel classes, which was restated recently in [1]. Using this inequality, we prove a strong unique continuation property of a second order elliptic partial differential equation that generalizes the result in [1] and [22].

1. INTRODUCTION

Let $1 \leq p < \infty$ and $\varphi : (0, \infty) \to (0, \infty)$. The **generalized Morrey space**, which was introduced by Nakai in [15] and denoted by $L^{p,\varphi}(\mathbb{R}^n) := L^{p,\varphi}$, is the collection of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ which satisfy

$$\|f\|_{L^{p,\varphi}} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{\varphi(r)} \int_{|x-y|<r} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

Note that $L^{p,\varphi}$ is a Banach space with norm $\|\cdot\|_{L^{p,\varphi}}$. If $\varphi(r) = 1$, then $L^{p,\varphi} = L^p$. If $\varphi(r) = r^n$, then $L^{p,\varphi} = L^\infty$. If $\varphi(r) = r^\lambda$ where $0 < \lambda < n$, then $L^{p,\varphi} = L^{p,\lambda}$ is the classical Morrey space introduced in [14].

We will assume the following conditions for φ which will be stated if needed:

(1) There exists $C > 0$ such that

$$s \leq t \Rightarrow \varphi(s) \leq C\varphi(t). \tag{1.1}$$

We say φ **almost increasing** if φ satisfies this condition.

(2) There exists $C > 0$ such that

$$s \leq t \Rightarrow \frac{\varphi(s)}{s^n} \geq C\frac{\varphi(t)}{t^n}. \tag{1.2}$$

We say $\varphi(t)t^{-n}$ **almost decreasing** if $\varphi(t)t^{-n}$ satisfies this condition.

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Observe that, if the function $\varphi(t)$ satisfies the conditions (1.1) and (1.2), then φ also satisfies the **doubling condition**, that is,

$$1 \leq \frac{s}{t} \leq 2 \Rightarrow \frac{1}{C} \leq \frac{\varphi(s)}{\varphi(t)} \leq C,$$

for some $C > 0$.

Let M be the **Hardy-Littlewood maximal operator**, that is,

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\, dy$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. The function $M(f)$ is called the **Hardy-Littlewood maximal function**. Notice that, for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ where $1 \leq p \leq \infty$, $M(f)(x)$ is finite for almost all $x \in \mathbb{R}^n$. Using Lebesgue Differentiation Theorem, we have

$$|f(x)| = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \leq \lim_{r\to 0} M(f)(x) = M(f)(x), \tag{1.3}$$

for every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and for almost all $x \in \mathbb{R}^n$. Furthermore, for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ where $1 \leq p \leq \infty$ and $0 < \gamma < 1$, the nonnegative function $w(x) = [M(f)(x)]^\gamma$ is an A_1 weight, that is,

$$M(w)(x) \leq C(n,\gamma) w(x).$$

These maximal operator properties can be found in [8, 20].

We will need the following theorem about the boundedness of the Hardy-Littlewood maximal operator on Morrey spaces $L^{p,\varphi}$.

Theorem 1.1 ([15, 18])**.** *Let φ satisfy conditions (1.1) and (1.2). If $1 \leq p < \infty$, then*

$$\|M(f)\|_{L^{p,\varphi}} \leq C(n,p)\|f\|_{L^{p,\varphi}},$$

for every $f \in L^{p,\varphi}$.

It should be noted that the proof of above theorem in [15] requires a condition about the integrability of $\varphi(t)t^{-(n+1)}$ over the interval (δ,∞) for every positive number δ. On the other hand, it requires only conditions (1.1) and (1.2) to show this theorem as in [18].

Let $1 \leq p < \infty$ and $0 < \alpha < n$. For $V \in L^p_{\text{loc}}(\mathbb{R}^n)$, we write

$$\eta_{\alpha,p}V(r) := \sup_{x\in\mathbb{R}^n} \left(\int\limits_{|x-y|<r} \frac{|V(y)|^p}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{1}{p}}, \quad r > 0.$$

We call $\eta_{\alpha,p}V$ the **Stummel p-modulus** of V. If $\eta_{\alpha,p}V(r)$ is finite for every $r > 0$, then $\eta_{\alpha,p}V(r)$ is nondecreasing on the set of positive real numbers and satisfies

$$\eta_{\alpha,p}V(2r) \leq C(n,\alpha)\, \eta_{\alpha,p}V(r), \quad r > 0.$$

The last inequality is known as the **doubling condition** for the Stummel $p-$modulus of V [21].

For each $0 < \alpha < n$ and $1 \leq p < \infty$, let

$$\tilde{S}_{\alpha,p} := \{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \,\text{for all}\, r > 0\}$$

and

$$S_{\alpha,p} := \left\{V \in L^p_{\text{loc}}(\mathbb{R}^n) : \eta_{\alpha,p}V(r) < \infty \,\text{ for all}\, r > 0 \,\text{and}\, \lim_{r\to 0} \eta_{\alpha,p}V(r) = 0\right\}.$$

The set $S_{\alpha,p}$ is called a **Stummel class**, while $\tilde{S}_{\alpha,p}$ is called a **bounded Stummel modulus class**. For $p = 1$, $S_{\alpha,1} := S_\alpha$ are the Stummel classes which were introduced in [5, 17]. We also write $\tilde{S}_{\alpha,1} := \tilde{S}_\alpha$ and $\eta_{\alpha,1} := \eta_\alpha$. It was shown in [21] that $\tilde{S}_{\alpha,p}$ contains $S_{\alpha,p}$ properly. These classes play an important role in studying the regularity theory of partial differential equations (see [1, 2, 5, 20, 22] for example).

In 1983, C. Fefferman [7] proved the following inequality:

$$\int_{\mathbb{R}^n} |u(x)|^2|V(x)|dx \leq C \|V\|_{L^{p,n-2p}} \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.4}$$

for $V \in L^{p,n-2p}$, where $1 < p \leq \frac{n}{2}$. Here V is the potential associated with the Schrödinger operator $L := -\Delta + V$. The inequality (1.4) is now known as **Fefferman's inequality**.

In 1990, Chiarenza and Frasca [3] generalized the inequality (1.4) by proving that

$$\int_{\mathbb{R}^n} |u(x)|^\alpha|V(x)|dx \leq C \|V\|_{L^{p,n-\alpha p}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha dx, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.5}$$

holds for $V \in L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \leq \frac{n}{\alpha}$. For the case $V \in \tilde{S}_2(\mathbb{R}^n)$, Zamboni [22] proved an inequality similar to (1.4), that is,

$$\int_{\mathbb{R}^n} |V(x)||u(x)|^2 dx \leq C\eta_2 f(r_0) \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \tag{1.6}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(u) \subseteq B_{r_0}$.

Recently, the inequality (1.6) is reproved in [1]. In this paper, we will generalize an inequality similar to (1.5) under the assumption that $V \in L^{p,\varphi}$, where φ satisfies the conditions (1.1), (1.2), (2.1) (see the condition (2.1) in Lemma 2.2). We will also prove an inequality similar to (1.6) by taking $V \in \tilde{S}_{\alpha,p}$ where $1 \leq \alpha \leq 2$.

It must be noted that $V \in \tilde{S}_{\alpha,p}$ is not contained in $L^{p,n-\alpha p}$, where $1 < \alpha < n$ and $1 < p \leq \frac{n}{\alpha}$. Indeed, if we define $V : \mathbb{R}^n \to \mathbb{R}$ by formula $V(y) := |y|^{-\frac{1}{p}}$, then $V \in \tilde{S}_{\alpha,p}$, but $V \notin L^{p,n-\alpha p}$. Therefore our result here (see Theorem 2.5) cannot be deduced from (1.5).

Let Ω be an open, bounded, and connected subset of $\mathbb{R}^n$. Recall that the **Sobolev space** $H^1(\Omega)$ is the set of all functions $u \in L^2(\Omega)$ for which $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$

for all $i = 1, \ldots, n$. Define the operator L on $H^1(\Omega)$ by

$$Lu := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + Vu \tag{1.7}$$

for $u \in H^1(\Omega)$, where a_{ij}, b_i $(i, j = 1, \ldots, n)$ and V are real valued measurable functions on Ω. Throughout this paper, we assume that the matrix $a(x) := (a_{ij}(x))$ is symmetric on Ω and that the ellipticity and boundedness conditions

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \tag{1.8}$$

hold for some $\lambda > 0$, for all $\xi \in \mathbb{R}^n$, and for almost all $x \in \Omega$. In addition, the functions b_i^2 $(i = 1, \ldots, n)$ and V in the equation (1.7) are assumed to belong to $L^{p,\varphi}$ (where φ satisfies conditions (1.1), (1.2), and (2.1)), or to $\tilde{S}_\alpha$ (where $1 \leq \alpha \leq 2$).

We say that $u \in H^1(\Omega)$ is a **weak solution** of the equation

$$Lu = 0 \tag{1.9}$$

if

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j} + \sum_{i=1}^{n} b_i\frac{\partial u}{\partial x_i}\psi + Vu\psi\right) dx = 0. \tag{1.10}$$

for all $\psi \in C_0^\infty(\Omega)$ (see the definition in [1, 22]). Note that, for the case $\alpha = 2$, the equation (1.9) was considered in [1, 22]. If we choose $b_i = 0$ for all $i = 1, \ldots, n$, then (1.9) becomes the Schrödinger equation. Observe that, if $b_i = 0$, $V \geq 0$ and $V \in L^\infty(\Omega)$, then the existence and uniqueness of the solution of (1.9) follows from the Lax-Milgram Theorem. However, we will not impose these restrictions and we will assume the existence of the solution.

Let $w \in L^1_{\text{loc}}(\Omega)$ and $w \geq 0$ in Ω. The function w is said to **vanish of infinite order** at $x_0 \in \Omega$ if

$$\lim_{r\to 0} \frac{1}{|B(x_0,r)|^k} \int_{B(x_0,r)} w(x)dx = 0, \qquad \forall k > 0.$$

The reader can examine that the real value function $w(x) = \exp\left(-|x|^{-1}\right)|x|^{-(n+1)}$ defined on $\mathbb{R}^n$ vanishes of infinite order at $x_0 = 0$.

The equation $Lu = 0$, which is given in (1.9), is said to have the **strong unique continuation property** in Ω if for every nonnegative solution u which vanishes of infinite order at some $x_0 \in \Omega$, then $u \equiv 0$ in $B(x_0, r)$ for some $r > 0$. See, for example, [9, 12, 13].

The following two lemmas tell us if a function vanishes of infinity order at some $x_0 \in \Omega$ and fulfills doubling integrability over some neighborhood of x_0, then the function must be identically to zero in the neighborhood.

Lemma 1.2 ([10]). *Let $w \in L^1_{\text{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$. Assume that there exists a constant $C > 0$ satisfying*

$$\int\limits_{B(x_0,r)} w(x)dx \le C \int\limits_{B\left(x_0,\frac{r}{2}\right)} w(x)dx.$$

If w vanishes of infinity order at x_0, then $w \equiv 0$ in $B(x_0, r)$.

Lemma 1.3. *Let $w \in L^1_{\text{loc}}(\Omega)$ and $B(x_0, r) \subseteq \Omega$, and $0 < \beta < 1$. Assume that there exists a constant $C > 0$ satisfying*

$$\int\limits_{B(x_0,r)} w^\beta(x)dx \le C \int\limits_{B\left(x_0,\frac{r}{2}\right)} w^\beta(x)dx.$$

If w vanishes of infinity order at x_0, then $w \equiv 0$ in $B(x_0, r)$.

Proof. According to the hypothesis, for every $j \in \mathbb{N}$ we have

$$\begin{aligned}\int_{B(x_0,r)} w^\beta(x)dx &\le C^1 \int_{B(x_0,2^{-1}r)} w^\beta(x)dx \\ &\le C^2 \int_{B(x_0,2^{-2}r)} w^\beta(x)dx \\ &\ \ \vdots \\ &\le C^j \int_{B(x_0,2^{-j}r)} w^\beta(x)dx.\end{aligned}$$

Hölder's inequality implies that

$$\left(\int_{B(x_0,r)} w^\beta(x)dx\right)^{\frac{1}{\beta}} \le (C^{\frac{1}{\beta}})^j |B(x_0, 2^{-j}r)|^{\frac{1}{\beta}} \frac{|B(x_0, 2^{-j}r)|^k}{|B(x_0, 2^{-j}r)|^{k+1}} \int_{B(x_0,2^{-j}r)} w(x)dx. \tag{1.11}$$

Now we choose $k > 0$ such that $C^{\frac{1}{\beta}} 2^{-k} = 1$. Then (1.11) gives

$$\left(\int_{B(x_0,r)} w^\beta(x)dx\right)^{\frac{1}{\beta}} \le (w_n r^n)^{\frac{1}{\beta}+k} (2^{-\frac{n}{\beta}})^j \frac{1}{|B(x_0, 2^{-j}r)|^{k+1}} \int_{B(x_0,2^{-j}r)} w(x)dx, \tag{1.12}$$

where w_n is the Lebesgue measure of unit ball in $\mathbb{R}^n$. Letting $j \to \infty$, we obtain from (1.12) that $w^\beta \equiv 0$ on $B(x_0, r)$. Therefore $w \equiv 0$ on $B(x_0, r)$. □

It will be shown in this paper that the equation $Lu = 0$, given by (1.9), has the strong unique continuation property in Ω. This property was studied by several authors, for example, Chiarenza and Garofalo in [3] when they discussed the Schrödinqer inequality of the form $Lu = -div(a\nabla u) + Vu \ge 0$, where the potential V belongs to Lorentz spaces $L^{\frac{n}{2},\infty}(\Omega)$. For the differential inequality of the form $|\Delta u| \le |V||u|$ where its potential also belong to $L^{\frac{n}{2}}(\Omega)$, see Jerison and Kenig [12]. Meanwhile, Garofalo and Lin [9] studied the equation (1.9) where the potentials are bounded by certain functions.

Our strong unique continuation result here is a consequence of Theorem 2.3 and Theorem 2.5 below. For the case $V \in \tilde{S}_\alpha$, where $\alpha = 2$, this property was obtained in [22] and restated recently in [1] (with the same proof).

2. Fefferman's Type Inequality

In this section, we prove the **Fefferman's inequality**, which we state in Theorem 2.3 and Theorem 2.5 below, and present some inequalities which are deduced from this inequality. We start with the case where the potential belongs to Morrey spaces.

Lemma 2.1. *Let φ satisfy the conditions (1.1) and (1.2). If $1 < \gamma < p$ and $V \in L^{p,\varphi}$, then $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in A_1 \cap L^{p,\varphi}$.*

Proof. According to our discussion above, $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in A_1$. Using Theorem 1.1, we have

$$\|[M(|V|^\gamma)]^{\frac{1}{\gamma}}\|_{L^{p,\varphi}} \leq \|M(|V|^\gamma)\|^{\frac{1}{\gamma}}_{L^{\frac{p}{\gamma},\varphi}} \leq \|V\|_{L^{p,\varphi}} < \infty.$$

Therefore $[M(|V|^\gamma)]^{\frac{1}{\gamma}} \in L^{p,\varphi}$. □

Lemma 2.2. *Let φ satisfy the conditions (1.1) and (1.2). Let $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and suppose that there exists a constant $C > 0$ such that for every $\delta > 0$,*

$$\int_\delta^\infty \frac{\varphi(t)}{t^{(n+1)-\frac{p}{2}(\alpha+1)}}\, dt \leq C\delta^{\frac{p}{2}(1-\alpha)}. \tag{2.1}$$

If $V \in L^{p,\varphi}$, then

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\, dy \leq C(n,\alpha,p)\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}[M(V)(x)]^{\frac{\alpha-1}{\alpha}}.$$

Proof. Let $\delta > 0$. Then

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\, dy = \int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy + \int_{|x-y|\geq\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy. \tag{2.2}$$

Using Lemma (a) in [11], we have

$$\int_{|x-y|<\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy \leq C(n)M(V)(x)\delta. \tag{2.3}$$

For the second term on the right hand side (2.2), let $q = n - \frac{p}{2}(\alpha+1)$, we use Hölder's inequality to obtain

$$\begin{aligned}\int_{|x-y|\geq\delta} \frac{|V(y)|}{|x-y|^{n-1}}\, dy &= \int_{|x-y|\geq\delta} \frac{|V(y)||x-y|^{\frac{q}{p}+1-n}}{|x-y|^{\frac{q}{p}}}\, dy \\ &\leq \left(\int_{|x-y|\geq\delta} \frac{|V(y)|^p}{|x-y|^q}\, dy\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{|x-y|\geq\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})} dy\right)^{\frac{p-1}{p}}.\end{aligned} \tag{2.4}$$

Note that

$$\int_{|x-y|\geq\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy = \sum_{k=0}^{\infty} \int_{2^k\delta\leq|x-y|<2^{k+1}\delta} \frac{|V(y)|^p}{|x-y|^q}\,dy$$
$$\leq C\|V\|^p_{L^{p,\varphi}} \int_\delta^\infty \frac{\varphi(t)}{t^{q+1}}\,dt$$
$$\leq C\|V\|^p_{L^{p,\varphi}} \delta^{n-p\alpha-q}. \tag{2.5}$$

Since $n + (\frac{q}{p} + 1 - n)(\frac{p}{p-1}) < 0$, we obtain

$$\int_{|x-y|\geq\delta} |x-y|^{(\frac{q}{p}+1-n)(\frac{p}{p-1})}dy = C(n,p,\alpha)\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}. \tag{2.6}$$

Introducing (2.5) and (2.6) in (2.4), we have

$$\int_{|x-y|\geq\delta} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \leq C\|V\|_{L^{p,\varphi}} \left(\delta^{n-p\alpha-q}\right)^{\frac{1}{p}} \left(\delta^{n+(\frac{q}{p}+1-n)(\frac{p}{p-1})}\right)^{\frac{p-1}{p}}$$
$$= C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha}. \tag{2.7}$$

From (2.7), (2.3) and (2.2), we get

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \leq CM(V)(x)\delta + C\|V\|_{L^{p,\varphi}}\delta^{1-\alpha} \tag{2.8}$$

For $\delta = \|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}[M(V)(x)]^{-\frac{1}{\alpha}}$, the inequality (2.8) becomes

$$\int_{\mathbb{R}^n} \frac{|V(y)|}{|x-y|^{n-1}}\,dy \leq C[M(V)(x)]^{1-\frac{1}{\alpha}}\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}} = C[M(V)(x)]^{\frac{\alpha-1}{\alpha}}\|V\|^{\frac{1}{\alpha}}_{L^{p,\varphi}}.$$

Thus, the lemma is proved. □

Now, we are ready to prove the Fefferman's inequality for case generalized Morrey spaces.

Theorem 2.3. *Let $1 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, and φ satisfy conditions (1.1), (1.2), (2.1). If $V \in L^{p,\varphi}$, then*

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)|dx \leq C\|V\|_{L^{p,\varphi}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha dx \tag{2.9}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let $1 < \gamma < p$ and $w := [M(|V|^\gamma)]^{\frac{1}{\gamma}}$. Then $w \in A_1 \cap L^{p,\varphi}$ according to the Lemma 2.1. First, we will show that (2.9) holds for w in place of V. For any $u \in C_0^\infty(\mathbb{R}^n)$, let B be a ball such that $u \in C_0^\infty(B)$. Consider the Poisson's equation

$$\begin{cases} -\Delta z = w & \text{on} \quad B \\ z = 0 & \text{on} \quad \partial B. \end{cases}$$

Let

$$z(x) = \int_B \Phi(x-y)w(y)\,dy, \quad x \in B,$$

be a solution of the Poisson's equation above, where Φ is the fundamental solution of Laplace's equation (see [6] pp. 22–23). Then

$$\begin{aligned} |\nabla z(x)| &\leq C \int_B |\nabla(\Phi(x-y))|\, w(y)\, dy \\ &\leq C \int_B \frac{w(y)}{|x-y|^{n-1}}\, dy \leq C \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-1}}\, dy \end{aligned} \tag{2.10}$$

where $C = C(n)$. From (2.10) and Lemma 2.2, we also have

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^\alpha w(x) dx &\leq \alpha \int_B |u(x)|^{\alpha-1} |\nabla u(x)| |\nabla z(x)| dx \\ &\leq C \int_B |u(x)|^{\alpha-1} |\nabla u(x)| \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-1}}\, dy dx \\ &\leq C \|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} \int_B |u(x)|^{\alpha-1} |\nabla u(x)| [M(w)(x)]^{\frac{\alpha-1}{\alpha}} dx. \end{aligned} \tag{2.11}$$

Hölder's inequality and Lemma (2.1) imply that

$$\begin{aligned} \int_B |u(x)|^{\alpha-1} |\nabla u(x)| [M(w)(x)]^{\frac{\alpha-1}{\alpha}}\, dx &\leq \left(\int_B |\nabla u(x)|^\alpha\, dx \right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\int_B |u(x)|^\alpha M(w)(x)\, dx \right)^{\frac{\alpha-1}{\alpha}} \\ &\leq C \left(\int_B |\nabla u(x)|^\alpha\, dx \right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\int_B |u(x)|^\alpha w(x)\, dx \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned} \tag{2.12}$$

Substituting (2.12) in (2.11), we obtain

$$\int_{\mathbb{R}^n} |u(x)|^\alpha |w(x)| dx \leq C \|w\|_{L^{p,\varphi}}^{\frac{1}{\alpha}} \left(\int_B |\nabla u(x)|^\alpha\, dx \right)^{\frac{1}{\alpha}} \left(\int_B |u(x)|^\alpha w(x)\, dx \right)^{\frac{\alpha-1}{\alpha}}.$$

Therefore

$$\int_{\mathbb{R}^n} |u(x)|^\alpha w(x) dx \leq C \|w\|_{L^{p,\varphi}} \int_B |\nabla u(x)|^\alpha\, dx.$$

By (1.3), we have $|V(x)| = [|V(x)|^\gamma]^{\frac{1}{\gamma}} \leq [M(|V(x)|^\gamma)]^{\frac{1}{\gamma}} = w(x)$. Hence, from Theorem 1.1 and Lemma 2.1, we conclude

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x)|^\alpha |V(x)| dx &\leq \int_{\mathbb{R}^n} |u(x)|^\alpha w(x) dx \\ &\leq C \|w\|_{L^{p,\varphi}} \int_B |\nabla u(x)|^\alpha\, dx \\ &\leq C \|V\|_{L^{p,\varphi}} \int_{\mathbb{R}^n} |\nabla u(x)|^\alpha\, dx. \end{aligned}$$

This completes the proof. $\square$

We already have shown in Theorem 2.3 that the Fefferman's inequality holds in generalized Morrey spaces. Next, we will prove this inequality for the case where the potential belongs to a Stummel class. We need the following lemma.

Lemma 2.4. *Let $1 < \alpha \le 2$ and $\alpha < n$. For any ball $B_0 \subset \mathbb{R}^n$, the following inequality holds:*

$$\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy \le \frac{C}{|x-z|^{\frac{n-1}{\alpha-1}-1}}, \quad x,z \in B_0, \quad x \ne z.$$

Proof. Let $r := \frac{1}{2}|x-z|$. Then

$$\begin{aligned}\int_{B_0} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy &\le \sum_{j=2}^{\infty}\int_{2^j r\le|x-y|<2^{j+1}r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy \\ &\quad + \int_{|x-y|<4r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy \\ &= I_1 + I_2. \end{aligned} \tag{2.13}$$

For I_1, we get

$$\begin{aligned} I_1 &= \sum_{j=2}^{\infty}\int_{2^j r\le|x-y|<2^{j+1}r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy \\ &\le \sum_{j=2}^{\infty}\frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}}\int_{2^j r\le|x-y|<2^{j+1}r} \frac{1}{|z-y|^{n-1}}\,dy. \end{aligned} \tag{2.14}$$

Note that, $2^j r \le |x-y| < 2^{j+1}r$ implies $2^j r \le |x-y| < 2r + |z-y|$. Therefore $2^{j-1}r \le 2^j r - 2r \le |z-y|$. Hence the inequality (2.14) becomes,

$$\begin{aligned} I_1 &\le \sum_{j=2}^{\infty}\frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}}\int_{2^j r\le|x-y|<2^{j+1}r} \frac{1}{|z-y|^{n-1}}\,dy \\ &\le C(n,\alpha)\sum_{j=2}^{\infty}\frac{1}{(2^j r)^{\frac{n-1}{\alpha-1}}}\frac{1}{(2^j r)^{n-1}}\int_{2^j r\le|x-y|<2^{j+1}r} 1\,dy \\ &\le C(n,\alpha)\frac{1}{(r)^{\frac{n-1}{\alpha-1}-1}}\sum_{j=2}^{\infty}\frac{1}{(2^j)^{\frac{n-1}{\alpha-1}-1}}. \end{aligned} \tag{2.15}$$

Since $\frac{n-1}{\alpha-1} - 1 > 0$, the last series in (2.15) is convergent. This gives us

$$I_1 \le C(n,\alpha)\frac{1}{(r)^{\frac{n-1}{\alpha-1}-1}} = \frac{C(n,\alpha)}{|x-z|^{\frac{n-1}{\alpha-1}-1}}. \tag{2.16}$$

For I_2, we obtain

$$\begin{aligned} I_2 &= \int_{|x-y|<r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy + \int_{r\le|x-y|<4r} \frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy \\ &\le C(n,\alpha)\frac{1}{r^{\frac{n-1}{\alpha-1}-1}} = \frac{C(n,\alpha)}{|x-z|^{\frac{n-1}{\alpha-1}-1}}. \end{aligned} \tag{2.17}$$

Combining (2.13), (2.16), and (2.17), the lemma is proved. □

The following theorem is the Fefferman's inequality where the potential belongs to a Stummel class.

Theorem 2.5. *Let* $1 \le p < \infty$, $1 \le \alpha \le 2$, *and* $\alpha < n$. *If* $V \in \tilde{S}_{\alpha,p}(\mathbb{R}^n)$, *then there exists a constant* $C := C(n,\alpha) > 0$ *such that*

$$\int\limits_{B(x_0,r_0)} |V(x)|^p |u(x)|^\alpha \, dx \le C[\eta_{\alpha,p}V(r_0)]^p \int\limits_{B(x_0,r_0)} |\nabla u(x)|^\alpha \, dx,$$

for every ball $B_0 := B(x_0,r_0) \subseteq \mathbb{R}^n$ *and* $u \in C_0^\infty(\mathbb{R}^n)$ *with* $\mathrm{supp}(u) \subseteq B_0$.

Proof. The proof is separated into two cases, namely $\alpha = 1$ and $1 < \alpha \le 2$. We first consider the case $\alpha = 1$. Using the well-known inequality

$$|u(x)| \le C \int_{B_0} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy \tag{2.18}$$

together with Fubini's theorem, we get

$$\begin{aligned}\int_{B_0} |u(x)||V(x)|^p \, dx &\le C \int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}} \, dxdy \\ &\le C \int_{B_0} |\nabla u(y)| \int_{B(y,2r_0)} \frac{|V(x)|^p}{|x-y|^{n-1}} \, dxdy.\end{aligned}$$

It follows from the last inequality and the doubling property of Stummel p-modulus of V that

$$\int_{B_0} |u(x)||V(x)|^p \, dx \le C\, \eta_{\alpha,p}V(r_0) \int_{B_0} |\nabla u(x)| \, dx,$$

as desired.

We now consider the case $1 < \alpha \le 2$. Using the inequality (2.18) and Hölder's inequality, we have

$$\begin{aligned}\int_{B_0} |u(x)|^\alpha |V(x)|^p \, dx &\le C \int_{B_0} |\nabla u(y)| \int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}} \, dx\, dy \\ &\le C \left(\int_{B_0} |\nabla u(y)|^\alpha\right)^{\frac{1}{\alpha}} \left(\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}} \, dy\right)^{\frac{\alpha-1}{\alpha}},\end{aligned} \tag{2.19}$$

where $F(y) := \displaystyle\int_{B_0} \frac{|u(x)|^{\alpha-1}|V(x)|^p}{|x-y|^{n-1}} \, dx$, $y \in B_0$. Applying Hölder's inequality again, we have

$$F(y) \le \left(\int_{B_0} \frac{|V(x)|^p}{|x-y|^{n-1}} \, dx\right)^{\frac{1}{\alpha}} \left(\int_{B_0} \frac{|u(z)|^\alpha |V(z)|^p}{|z-y|^{n-1}} \, dz\right)^{\frac{\alpha-1}{\alpha}},$$

so that

$$\begin{aligned}\int_{B_0} F(y)^{\frac{\alpha}{\alpha-1}}\,dy &\le \int_{B_0}\left(\int_{B_0}\frac{|V(x)|^p}{|x-y|^{n-1}}\,dx\right)^{\frac{1}{\alpha-1}}\int_{B_0}\frac{|u(z)|^\alpha|V(z)|^p}{|z-y|^{n-1}}\,dz\,dy\\ &=\int_{B_0}|u(z)|^\alpha|V(z)|^pG(z)\,dz,\end{aligned}\tag{2.20}$$

where $G(z) := \int_{B_0}\left(\int_{B_0}\frac{|V(x)|^p}{|x-y|^{n-1}|z-y|^{(n-1)(\alpha-1)}}\,dx\right)^{\frac{1}{\alpha-1}}\,dy,\ z\in B_0$. By virtue of Minkowski's integral inequality (or Fubini's theorem for $\alpha = 2$), we see that

$$G(z)^{\alpha-1} \le \int_{B_0}|V(x)|^p\left(\int_{B_0}\frac{1}{|x-y|^{\frac{n-1}{\alpha-1}}|z-y|^{n-1}}\,dy\right)^{\alpha-1}\,dx. \tag{2.21}$$

Combining (2.21), doubling property of Stummel p-modulus of V, and the inequality in Lemma 2.1, we obtain

$$G(z) \le C\left(\int_{B_0}\frac{|V(x)|^p}{|x-z|^{n-\alpha}}\,dx\right)^{\frac{1}{\alpha-1}} \le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha-1}}. \tag{2.22}$$

Now, (2.20) and (2.22) give

$$\int_{B_0}|F(y)|^{\frac{\alpha}{\alpha-1}}\,dy \le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha-1}}\int_{B_0}|u(x)|^\alpha|V(x)|^p\,dx. \tag{2.23}$$

Therefore, from (2.19) and (2.23), we get

$$\begin{aligned}&\int_{B_0}|u(x)|^\alpha|V(x)|^p\,dx\\ &\quad\le C[\eta_{\alpha,p}V(r_0)]^{\frac{p}{\alpha}}\left(\int_{B_0}|\nabla u(x)|^\alpha\,dx\right)^{\frac{1}{\alpha}}\left(\int_{B_0}|u(x)|^\alpha|V(x)|^p\,dx\right)^{\frac{\alpha-1}{\alpha}}.\end{aligned}\tag{2.24}$$

Dividing both sides by the third term of the right-hand side of (2.24), we get the Fefferman's inequality. □

Let B be an open ball in $\mathbb{R}^n$. If u has weak gradient ∇u in B and u is integrable over B, then by the sub-representation inequality we have

$$|u(x)-u_B| \le c(n)\int_B\frac{|\nabla u(y)|}{|x-y|^{n-1}}dy,\quad x\in B, \tag{2.25}$$

where $u_B := \frac{1}{|B|}\int_B u(y)dy$. Using the inequality (2.25) and the method in the proof of the previous theorem, we obtain the following result.

Theorem 2.6. *Let $1 \le p < \infty$, $1 \le \alpha \le 2$, and $\alpha < n$. Suppose that u has weak gradient ∇u in $B_0 := B(x_0, r_0) \subseteq \mathbb{R}^n$ and that u is integrable over B_0. If*

$V \in \tilde{S}_{\alpha,p}$, then

$$\int_{B_0} |u(x) - u_{B(x_0,r_0)}|^\alpha |V(x)|^p dx \le C\,[\eta_{\alpha,p} V(r_0)]^p \int_{B_0} |\nabla u(x)|^\alpha dx,$$

where $C := C(n, \alpha)$.

Remark 2.7. Note that the case $\alpha = 2$ is exactly the Corollary 4.4 in [1].

3. Unique Continuation Property

In this section, we assume the functions b_i^2 and V in equation (1.9), that is $Lu = 0$, belong to $\tilde{S}_\alpha$ where $1 \le \alpha \le 2$, or to $L^{p,\varphi}$ where $1 < \alpha \le 2$, $1 < p < \frac{n}{\alpha}$, and φ satisfies (1.1), (1.2), and (2.1). We will give an application of Theorem 2.3 and Theorem 2.5 in proving the strong unique continuation result for the equation $Lu = 0$. Precisely, we use Theorem 2.3 or Theorem 2.5 in proving Theorem 3.1 below and we deduce that $\log(u+\delta) \in BMO_\alpha(B)$ (see this definition below) for every $\delta > 0$, where $B \subseteq \Omega$ is a ball with radius less than or equal to 1.

A locally integrable function f on $\mathbb{R}^n$ is said to be of **bounded mean oscillation** on ball $B \subseteq \mathbb{R}^n$ if there is a constant $C > 0$ such that for every ball $B' \subseteq B$,

$$\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}| dy \le C.$$

We write $f \in BMO(B)$ if f is of bounded mean oscillation on B. Moreover, if $1 \le \alpha < \infty$ and there is a constant $C > 0$ such that for every ball $B' \subseteq B$,

$$\left(\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}|^\alpha dy \right)^{\frac{1}{\alpha}} \le C,$$

we write $f \in BMO_\alpha(B)$.

Now, let $1 \le \beta < \alpha < \infty$ and $f \in BMO_\alpha(B)$. Given a ball $B' \subseteq B$. Hölder's inequality implies

$$\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}|^\beta dy \le \frac{1}{|B'|} \left(\int_{B'} |f(y) - f_{B'}|^\alpha dy \right)^{\frac{\beta}{\alpha}} \left(\int_{B'} 1\, dy \right)^{1-\frac{\beta}{\alpha}}.$$

Therefore

$$\left(\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}|^\beta dy \right)^{\frac{1}{\beta}} \le \left(\frac{1}{|B'|} \int_{B'} |f(y) - f_{B'}|^\alpha dy \right)^{\frac{1}{\alpha}} \le C.$$

This tells us that $BMO_\alpha(B) \subseteq BMO_\beta(B)$.

Theorem 3.1. *Let $u \ge 0$ be a weak solution of $Lu = 0$ and $B(x, 2r) \subseteq \Omega$ where $r \le 1$. Then there exists a constant $C > 0$ such that for every $\delta > 0$ we have*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |\log(u+\delta) - \log(u+\delta)_B|^\alpha \, dy \le C.$$

Proof. Let $\psi \in C_0^\infty(B(x,2r))$, $0 \le \psi \le 1$, $|\nabla\psi| \le C_1 r^{-1}$, and $\psi := 1$ on $B(x,r)$. Using (1.8) and the weak solution definition (1.10), we have

$$\lambda \int_\Omega \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \le (\alpha+1)\int_\Omega \langle a\nabla u, \nabla\psi\rangle \frac{\psi^\alpha}{(u+\delta)} + \sum_{i=1}^n \int_\Omega b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u+\delta)} + \int_\Omega V\psi^{\alpha+1}. \tag{3.1}$$

Since $\mathrm{supp}(\psi) \subseteq B(x,2r)$, the inequality (3.1) reduces to

$$\lambda \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} \le (\alpha+1)\int_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^\alpha}{(u+\delta)} + \sum_{i=1}^n \int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u+\delta)} + \int_{B(x,2r)} V\psi^{\alpha+1}. \tag{3.2}$$

We will estimate the first term integral on the right hand side (3.2). According to (1.8), we have

$$|\langle a\nabla u, \nabla\psi\rangle| \le \lambda^{-1}|\nabla u||\nabla\psi|. \tag{3.3}$$

Combining the Young's inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$ for every $\epsilon > 0$ $(a,b>0)$ and the inequality (3.3), we have for every $\epsilon > 0$

$$\begin{aligned}(\alpha+1)\int_{B(x,2r)} \langle a\nabla u, \nabla\psi\rangle \frac{\psi^\alpha}{(u+\delta)} &\le \epsilon\lambda^{-1}(\alpha+1)\int_{B(x,2r)} \frac{|\nabla u|^2}{(u+\delta)^2}\psi^{2\alpha} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int_{B(x,2r)} |\nabla\psi|^2 \\ &\le \epsilon\lambda^{-1}(\alpha+1)\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{\lambda^{-1}(\alpha+1)}{4\epsilon}\int_{B(x,2r)} |\nabla\psi|^2. \end{aligned} \tag{3.4}$$

To estimate the second term in (3.2), we use Hölder's inequality, Young's inequality and Theorem 2.3 or Theorem 2.5, to obtain

$$\begin{aligned}\int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i}\frac{\psi^{\alpha+1}}{(u+\delta)} &\le \int_{B(x,2r)} |b_i||\nabla u|\frac{\psi^{\alpha+1}}{(u+\delta)} \\ &\le \left(\int_{B(x,2r)} \frac{|\nabla u|^2}{(u+\delta)^2}\psi^{\alpha+1}\right)^{\frac12}\left(\int_{B(x,2r)} b_i^2\psi^{\alpha+1}\right)^{\frac12} \\ &\le \frac{\epsilon}{n}\int_{B(x,2r)} \frac{|\nabla u|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}\int_{B(x,2r)} b_i^2\psi^\alpha \\ &\le \frac{\epsilon}{n}\int_{B(x,2r)} \frac{|\nabla u|^2}{(u+\delta)^2}\psi^{\alpha+1} + \frac{1}{4n\epsilon}C_2^i\int_{B(x,2r)} |\nabla\psi|^\alpha. \end{aligned} \tag{3.5}$$

for every $i = 1, \ldots, n$, where the constant C_2^i depends on n, α, $\|b_i^2\|_{L^{p,\varphi}}$, or $\eta_\alpha b_i^2(r_0)$. From (3.5), we have

$$\sum_{i=1}^{n} \int_{B(x,2r)} b_i \frac{\partial u}{\partial x_i} \frac{\psi^{\alpha+1}}{(u+\delta)} \leq \epsilon \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2} \psi^{\alpha+1} + \frac{1}{4\epsilon} C_3 \int_{B(x,2r)} |\nabla\psi|^\alpha, \tag{3.6}$$

where C_3 depends on $\max_i\{C_2^i\}$. The estimation of the last term in (3.2) is

$$\int_{B(x,2r)} V\psi^{\alpha+1} \leq \int_{B(x,2r)} V\psi^\alpha \leq C_4 \int_{B(x,2r)} |\nabla\psi|^\alpha, \tag{3.7}$$

where the constant C_4 depends on n, α, and $\|V\|_{L^{p,\varphi}}$, or $\eta_\alpha V(r_0)$. Now, choose $\epsilon := \frac{1}{2}\frac{\lambda^2}{(\alpha+1)+1}$. Introducing (3.4), (3.6), and (3.7) in (3.2), we get

$$\int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2} \psi^{\alpha+1} \leq C_5 \int_{B(x,2r)} |\nabla\psi|^2 + C_6 \int_{B(x,2r)} |\nabla\psi|^\alpha, \tag{3.8}$$

where the constant C_5 depends on α and λ, while the constant C_6 depends on C_3 and C_4. Therefore, (3.8) implies

$$\begin{aligned}\int_{B(x,r)} |\nabla \log(u+\delta)|^2 &\leq \int_{B(x,2r)} \frac{|\nabla(u+\delta)|^2}{(u+\delta)^2} \psi^{\alpha+1} \\ &\leq C_5 \int_{B(x,2r)} |\nabla\psi|^2 + C_6 \int_{B(x,2r)} |\nabla\psi|^\alpha \\ &\leq C\left(r^{-2}r^n + r^{-\alpha}r^n\right) = Cr^{-2}r^n.\end{aligned}$$

The last constant C depends on C_1, C_5, and C_6. From Hölder's inequality,

$$\left(\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha\right)^{\frac{2}{\alpha}} \leq \frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^2 \leq Cr^{-2},$$

whence

$$\frac{1}{r^n}\int_{B(x,r)} |\nabla \log(u+\delta)|^\alpha \leq Cr^{-\alpha}. \tag{3.9}$$

By using Poincaré's inequality together with the inequality (3.9), the proposition is proved. □

By virtue of Theorem 3.1 and the previous discussion, we have the following corollary.

Corollary 3.2. *Let $u \geq 0$ be a weak solution of $Lu = 0$ and $B(x, 2r) \subseteq \Omega$ where $r \leq 1$. Then, for every $\delta > 0$, $\log(u + \delta) \in BMO_\alpha(B(x, r))$.*

We recall the celebrated theorem which is due to John-Nirenberg. If $f \in BMO(B)$, then there exist $\beta > 0$ and $M > 0$ such that for every ball $B' \subseteq B$

$$\int_{B'} \exp(\beta|f(x) - f_{B'}|)\, dx \leq M|B'|$$

We refer to [16] for more detail information about this **John-Nirenberg Theorem**.

Let $\log(u) \in BMO(B)$ for an appropriate function $u \geq 0$, where $B = B(x,r)$. By John-Nirenberg Theorem, there exist $\beta > 0$ and $M > 0$ such that

$$\left(\int_B \exp(\beta|\log(u) - \log(u)_B|)\, dy\right)^2 \leq M^2|B|^2. \tag{3.10}$$

Assume that $\beta < 1$. Using (3.10), we compute

$$\begin{aligned}
&\left(\int_B u^\beta dy\right)\left(\int_B u^{-\beta} dy\right)\\
&= \left(\int_B \exp(\beta\log(u))dy\right)\left(\int_B \exp(-\beta\log(u))dy\right)\\
&= \left(\int_B \exp(\beta(\log(u) - \log(u)_B))dy\right)\left(\int_B \exp(-\beta(\log(u) - \log(u)_B))dy\right)\\
&\leq \left(\int_B \exp(\beta|\log(u) - \log(u)_B|)\, dy\right)^2 \leq M^2|B|^2,
\end{aligned}$$

which gives

$$\left(\int_B u^{-\beta} dy\right)^{\frac{1}{2}} \leq M|B|\left(\int_B u^\beta dy\right)^{-\frac{1}{2}} \leq M|B|\left(\int_{B(x,2r)} u^\beta dy\right)^{-\frac{1}{2}}. \tag{3.11}$$

Applying Hölder's inequality and (3.11), we obtain

$$\begin{aligned}
|B| &\leq \int_B u^{\frac{\beta}{2}} u^{-\frac{\beta}{2}} dy \leq \left(\int_B u^\beta dy\right)^{\frac{1}{2}}\left(\int_B u^{-\beta} dy\right)^{\frac{1}{2}}\\
&\leq M|B|\left(\int_B u^\beta dy\right)^{\frac{1}{2}}\left(\int_{B(x,2r)} u^\beta dy\right)^{-\frac{1}{2}}.
\end{aligned} \tag{3.12}$$

From (3.12), we get

$$\int_{B(x,2r)} u^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u^\beta dy.$$

The last inequality together with Lemma 1.3 tell us that if u vanishes of infinity order at x, then $u \equiv 0$ in $B(x,2r)$.

For the case $\beta \geq 1$, we obtain from the inequality (3.10) that

$$\left(\int_B \exp(|\log(u) - \log(u)_B|)\, dy\right)^2 \leq \left(\int_B \exp(\beta|\log(u) - \log(u)_B|)\, dy\right)^2 \leq M^2|B|^2.$$

Processing the last inequality with previously method, we get

$$\int_{B(x,2r)} u\, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u\, dy.$$

According to Lemma 1.2, if u vanishes of infinity order at x, then $u \equiv 0$ in $B(x,2r)$.

Corollary 3.3. *The equation $Lu = 0$ has the strong unique continuation property in Ω.*

Proof. Given $x \in \Omega$ and let $B := B(x,r)$ be a ball where $B(x,2r) \subseteq \Omega$ and $r \leq 1$. Let $\{\delta_j\}$ be a sequence of real numbers in $(0,1)$ which converges to 0. From Corollary 3.2, we get $\log(u+\delta_j) \in BMO_\alpha(B)$. Therefore $\log(u+\delta_j) \in BMO(B)$. According to our previous discussion, there exists a constant $M > 0$ such that we have two cases:

$$\int_{B(x,2r)} u^\beta dy \leq \int_{B(x,2r)} (u+\delta_j)^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} (u+\delta_j)^\beta dy,$$

where $0 < \beta < 1$, or,

$$\int_{B(x,2r)} u\, dy \leq \int_{B(x,2r)} (u+\delta_j)\, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} (u+\delta_j)\, dy.$$

In both cases, letting $j \to \infty$, we obtain

$$\int_{B(x,2r)} u^\beta dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u^\beta dy,$$

or,

$$\int_{B(x,2r)} u\, dy \leq M^{\frac{1}{2}} \int_{B(x,r)} u\, dy.$$

Therefore $u \equiv 0$ in $B(x,2r)$ if u vanishes of infinity order at x. □

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