



Research Article

A Partial Solution To An Open Problem

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Abstract

Batkunde et al. [Acta Univ. M. Belii Ser. Math., 2013] have defined a multilinear n -functional on l^p . Regarding the n -functional F_Y on $(l^p, \|\cdot\|, \dots, \|\cdot\|_p)$, they want to compute the exact norm of F_Y , especially for $p \neq 2$. In this paper, we deal with a partial solution to an open problem given in their paper.

Keywords: Inner product spaces, n -normed spaces, bounded multilinear n -functional, p -summable sequences.

1. Introduction

Let $n \geq 2$ be an integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties:

- i. $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent,
- ii. $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation,
- iii. $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$, for any $\alpha \in \mathbb{R}$,
- iv. $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called an n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space (Gunawan & Mashadi 2001).

Recent results and related topics may be found in (Gahler 1965; Batkunde et al. 2013; Gunawan & Mashadi 2001; Gunawan et al. 2005; Gozali et al. 2010; Pangalela & Gunawan 2013; Gunawan 2002; Gozali et al. 2010; Gunawan 2001; Milicic 1993).

Throughout the paper, we have focused on the space of p -summable sequences, denoted by l^p , where $1 \leq p < \infty$.

Recall that a sequence $u = (u_k)_{k=1}^\infty$ (of real numbers)

belongs l^p space if $\|u\|_p := \left(\sum_{k=1}^\infty |u_k|^p \right)^{\frac{1}{p}} < \infty$. It is known

that the dual space of l^p is l^q where $\frac{1}{p} + \frac{1}{q} = 1$. Let

$(X, \|\cdot, \dots, \cdot\|)$ be a real n -normed space and $f : X \rightarrow \mathbb{R}$ be a linear functional on X . Several n -norms on l^p , which

can be seen in (Batkunde et al. 2013), are given as follows:

If $(X, \|\cdot\|)$ is a normed space and X' is its dual (consisting of bounded linear functionals on X), the following function defines an n -norm on X :

$$\|x_1, \dots, x_n\|_p^G := \sup_{f_i \in X', \|f_i\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}. \quad (1.1)$$

Using the formula (1.1), l^p may be equipped with the following n -norm:

$$\|x_1, \dots, x_n\|_p^G := \sup_{y_i \in l^q, \|y_i\|_q \leq 1} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \cdots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \cdots & \sum_k x_{nk} y_{nk} \end{vmatrix}, \quad (1.2)$$

where q denotes the dual exponent of p . There is another formula of n -norm which can be defined on l^p (Batkunde et al. 2013), namely

$$\|x_1, \dots, x_n\|_p := \left(\frac{1}{n!} \sum_{k_1} \cdots \sum_{k_n} \left\| \begin{vmatrix} x_{1k_1} & \cdots & x_{1k_n} \\ \vdots & \ddots & \vdots \\ x_{nk_1} & \cdots & x_{nk_n} \end{vmatrix} \right\|^p \right)^{\frac{1}{p}}, \quad (1.3)$$

where $x_i = (x_{ik})_{k=1}^{\infty}$, $i=1,2,\dots,n$. As shown in (Gunawan 2002), the two n -norms are equivalent:

$$(n!)^{\frac{1}{p}-1} \|x_1, \dots, x_n\|_p \leq \|x_1, \dots, x_n\|_p^G \leq (n!)^{\frac{1}{p}} \|x_1, \dots, x_n\|_p.$$

On l^2 , both n -norms coincide with the standard n -norm given by $\|x_1, \dots, x_n\|_2 := \sqrt{\det(\langle x_i, x_j \rangle)}$. Next observe that the determinant on the right hand side of (1.2) can be rewritten as

$$\frac{1}{n!} \sum_{j_1} \dots \sum_{j_n} \begin{vmatrix} x_{1j_1} & \dots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \dots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \dots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \dots & y_{nj_n} \end{vmatrix}. \quad (1.4)$$

By Hölder's inequality, it is dominated by $\|x_1, \dots, x_n\|_p \|y_1, \dots, y_n\|_q$. Another n -norm on l^p defined by Batkunda et al. (2013), namely

$$\|x_1, \dots, x_n\|_p^I := \sup_{y_i \in l^q, \|y_1, \dots, y_n\|_q^I \leq 1} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \dots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \dots & \sum_k x_{nk} y_{nk} \end{vmatrix}. \quad (1.5)$$

As can be seen in (Batkunda et al. 2013), the three n -norms on l^p given in (1.2), (1.3) and (1.5) are equivalent:

$$\|x_1, \dots, x_n\|_p^I \leq \|x_1, \dots, x_n\|_p \leq (n!)^{\frac{1}{p-1}} \|x_1, \dots, x_n\|_p^G \leq n! \|x_1, \dots, x_n\|_p^I. \quad (1.6)$$

On a normed space $(X, \|\cdot\|)$, the functional $g: X^2 \rightarrow \mathbb{R}$

defined by the formula $g(x, y) := \frac{\|x\|}{2} (\lambda_+(x, y) + \lambda_-(x, y))$,

where $\lambda_{\pm}(x, y) := \lim_{t \rightarrow \pm 0} t^{-1} (\|x + ty\| - \|x\|)$, satisfies the following properties:

- i. $g(x, x) = \|x\|^2$ for all $x \in X$,
- ii. $g(\alpha x, \beta y) = \alpha \beta g(x, y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$,
- iii. $g(x, x+y) = \|x\|^2 + g(x, y)$ for all $x, y \in X$,

iv. $|g(x, y)| \leq \|x\| \|y\|$ for all $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear in $y \in X$, it is called a semi-inner product on X (Milicic 1993).

The functional

$$g(x, y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \text{sgn}(x_k) y_k, \quad x = (x_k), \quad y = (y_k) \in l^p \quad (1.7)$$

defines a semi-inner product on the space l^p , for $1 \leq p < \infty$, where $\|\cdot\|_p$ is the usual norm on l^p . Using a semi-inner product g , one may define the notion of orthogonality on X . In particular, it can be defined

$$x \perp_g y \Leftrightarrow g(x, y) = 0. \quad (1.8)$$

Note that since g is in general not commutative, $x \perp_g y$ does not imply that $y \perp_g x$ (Milicic 1993).

2. Bounded Multilinear n -Functionals on l^p

A multilinear n -functional on a real vector space X is a mapping $F: X^n \rightarrow \mathbb{R}$ which is linear in each variable. A multilinear n -functional F is bounded on an n -normed space $(X, \|\cdot\|, \dots, \|\cdot\|)$ if and only if there exists $K > 0$ such that

$$|F(x_1, \dots, x_n)| \leq K \|x_1, \dots, x_n\| \quad (2.1)$$

for every $x_1, \dots, x_n \in X$. Note that for a bounded multilinear n -functional F on an n -normed space $(X, \|\cdot\|, \dots, \|\cdot\|)$, we have $F(x_1, \dots, x_n) = 0$ when $x_1, \dots, x_n$ are linearly dependent (Batkunda et al. 2013).

If F is a bounded multilinear n -functional on an n -normed space $(X, \|\cdot\|, \dots, \|\cdot\|)$, then F is antisymmetric, that is

$$F(x_1, \dots, x_n) = \text{sgn}(\sigma) F(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for any $x_1, \dots, x_n \in X$ and any permutation σ of $(1, \dots, n)$. Here $\text{sgn}(\sigma) = 1$ if σ is an even permutation and $\text{sgn}(\sigma) = -1$ if σ is an odd permutation. These properties do not hold for bounded multilinear n -functionals on a normed space $(X, \|\cdot\|)$ (Batkunde et al. 2013).

The set X' of all bounded multilinear n -functionals on $(X, \|\cdot\|, \dots, \|\cdot\|)$ forms a vector space. A bounded multilinear n -functional F is defined

$$\|F\| := \inf \{K > 0 : (2.1) \text{ holds}\},$$

or equivalently

$$\|F\| := \sup \{|F(x_1, \dots, x_n)| : \|x_1, \dots, x_n\| \leq 1\}.$$

This formula defines a norm on X' .

Let $Y := \{y_1, \dots, y_n\}$ in l^q , where q is the dual exponent of p . Batkunde et al. (2013) defined the following multilinear n -functional on l^p where $1 \leq p < \infty$:

$$F_Y(x_1, \dots, x_n) := \frac{1}{n!} \sum_{j_1} \dots \sum_{j_n} \begin{vmatrix} x_{1j_1} & \dots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \dots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \dots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \dots & y_{nj_n} \end{vmatrix}, \quad (2.2)$$

for $x_1, \dots, x_n \in l^p$. From the definition of the multilinear n -functional F_Y in (2.2), clearly, if Y is linearly dependent set, then $F_Y(x_1, \dots, x_n) = 0$. For this purpose, we separate this case and we assume that if Y is linearly dependent set, then $F_Y(x_1, \dots, x_n) := 0$ and if $Y := \{y_1, \dots, y_n\}$ is linearly independent set in l^q , then the multilinear n -functional F_Y on l^p is defined as in (2.2).

Clearly F_Y is linear in each variable. Further,

$$|F_Y(x_1, \dots, x_n)| \leq \|x_1, \dots, x_n\|_p \|y_1, \dots, y_n\|_q \text{ and so } F_Y \text{ is bounded}$$

on $(l^p, \|\cdot\|, \dots, \|\cdot\|_p)$ with $\|F_Y\| \leq \|y_1, \dots, y_n\|_q$. For $p = 2$, the following fact is obtained (Batkunde et al. 2013).

Fact 2.3. (Batkunde et al. 2013) Consider the n -normed space $(l^2, \|\cdot\|, \dots, \|\cdot\|_2)$. For fixed linearly independent set $Y := \{y_1, \dots, y_n\}$ in l^2 , let F_Y be the multilinear n -functional defined as in (2.2). Then F_Y is bounded on $(l^2, \|\cdot\|, \dots, \|\cdot\|_2)$ with $\|F_Y\| = \|y_1, \dots, y_n\|_2$.

Proof. From the inequality

$$|F_Y(x_1, \dots, x_n)| \leq \|x_1, \dots, x_n\|_2 \|y_1, \dots, y_n\|_2$$

we see that F_Y is bounded with

$$\|F_Y\| \leq \|y_1, \dots, y_n\|_2.$$

Since $Y := \{y_1, \dots, y_n\}$ is a linearly independent set in l^2 , we can choose $x_i := \frac{y_i}{\sqrt[n]{\|y_1, \dots, y_n\|_2}}$, $i = 1, \dots, n$. If

$$x_i := \frac{y_i}{\sqrt[n]{\|y_1, \dots, y_n\|_2}}, \quad i = 1, \dots, n, \quad \text{then} \quad \|x_1, \dots, x_n\|_2 = 1 \quad \text{and}$$

$$F_Y(x_1, \dots, x_n) = \|y_1, \dots, y_n\|_2. \quad \text{Hence, this conclude that} \\ \|F_Y\| = \|y_1, \dots, y_n\|_2.$$

3. Main Results

Regarding the n -functional F_Y on $(l^p, \|\cdot\|, \dots, \|\cdot\|_p)$, an open problem was given in (Batkunde et al. 2013). In this paper, we give a partial solution for this open problem.

Open Problem. Compute the exact norm of F_Y in (2.2), especially for $p \neq 2$.

The proof is not easy. If an exact solution can not be found, then it may be possible to obtain equivalence of norms such that

$$\frac{1}{n!} \|y_1, \dots, y_n\|_p \leq \|F_Y\| \leq \|y_1, \dots, y_n\|_p.$$

Proof. Recall that (1.4) can be obtained from the determinant given on the right hand side of the equation (1.2). Then the multilinear n -functional on l^p can be rewritten as:

$$F_Y(x_1, \dots, x_n) := \begin{vmatrix} \sum_k x_{1k} y_{1k} & \cdots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \cdots & \sum_k x_{nk} y_{nk} \end{vmatrix}.$$

It is known from above that F_Y is bounded on $(l^p, \|\cdot\|_p)$

with $|F_Y(x_1, \dots, x_n)| \leq \|y_1, \dots, y_n\|_q$. To show the left part of the inequality, choose the linearly independent set

$Y := \{y_1, \dots, y_n\}$ be a left g -orthogonal in l^q such that

$y_i \perp_g y_j$ with $i < j$ for $1 \leq i, j \leq n$. Next, if we take

$$z_{i_k} := |y_{i_k}|^{q-1} \operatorname{sgn}(y_{i_k}), \quad 1 \leq i, k \leq n \quad \text{and}$$

$$x_i := \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_i\|_q^q} |y_{ik}|^{q-1} \operatorname{sgn}(y_{ik}), \quad i = 1, \dots, n, \quad k \in \square \quad \text{and}$$

$y_i \neq 0$ for each $i \in \square$, then

$$\begin{aligned} & \|x_1, \dots, x_n\|_p \\ &= \left\| \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \begin{vmatrix} \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_{1j_1}\|_q^q} |y_{1j_1}|^{q-1} \operatorname{sgn}(y_{1j_1}) & \cdots & \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_{1j_n}\|_q^q} |y_{1j_n}|^{q-1} \operatorname{sgn}(y_{1j_n}) \\ \vdots & \ddots & \vdots \\ \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_{nj_1}\|_q^q} |y_{nj_1}|^{q-1} \operatorname{sgn}(y_{nj_1}) & \cdots & \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_{nj_n}\|_q^q} |y_{nj_n}|^{q-1} \operatorname{sgn}(y_{nj_n}) \end{vmatrix} \right\|_p^{\frac{1}{p}} \\ &= \frac{\|y_1, \dots, y_n\|_q}{n! \prod_{i=1}^n \|y_i\|_q^q} \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \begin{vmatrix} |y_{1j_1}|^{q-1} \operatorname{sgn}(y_{1j_1}) & \cdots & |y_{1j_n}|^{q-1} \operatorname{sgn}(y_{1j_n}) \\ \vdots & \ddots & \vdots \\ |y_{nj_1}|^{q-1} \operatorname{sgn}(y_{nj_1}) & \cdots & |y_{nj_n}|^{q-1} \operatorname{sgn}(y_{nj_n}) \end{vmatrix} \right]^{\frac{1}{p}} \\ &= \frac{\|y_1, \dots, y_n\|_q}{(n!)^{\frac{1}{p}} \prod_{i=1}^n \|y_i\|_q^q} \|z_1, \dots, z_n\|_p \\ &= \frac{\|y_1, \dots, y_n\|_q \|z_1, \dots, z_n\|_p}{(n!)^{\frac{1}{p}} \prod_{i=1}^n \|y_i\|_q^q (n!)^{\frac{1}{q}} \prod_{i=1}^n \|y_i\|_q^{q-1}} \end{aligned}$$

$$\begin{aligned} &= \frac{\|y_1, \dots, y_n\|_q \|z_1, \dots, z_n\|_p}{(n!)^{\frac{1}{p}} \prod_{i=1}^n \|y_i\|_q^q (n!)^{\frac{1}{q}} \prod_{i=1}^n \left[\sum_j |y_{ij}|^q \right]^{\frac{q-1}{q}}} \\ &= \frac{\|y_1, \dots, y_n\|_q \|z_1, \dots, z_n\|_p}{(n!)^{\frac{1}{p}} \prod_{i=1}^n \|y_i\|_q^q (n!)^{\frac{1}{q}} \prod_{i=1}^n \left[\sum_j |y_{ij}|^{q-1} \operatorname{sgn}(y_{ij}) \right]^{\frac{1}{p}}} \\ &= \frac{\|y_1, \dots, y_n\|_q}{(n!)^{\frac{1}{p}} \prod_{i=1}^n \|y_i\|_q^q} \frac{\|z_1, \dots, z_n\|_p}{(n!)^{\frac{1}{q}} \prod_{i=1}^n \|z_i\|_p} \\ &\leq 1, \end{aligned}$$

since $\|y_1, \dots, y_n\|_q \leq (n!)^{\frac{1}{p}} \|y_1\|_q \cdots \|y_n\|_q$ and

$$\|z_1, \dots, z_n\|_p \leq (n!)^{\frac{1}{q}} \|z_1\|_p \cdots \|z_n\|_p \quad [\text{see, (Gunawan 2001)}].$$

Hence

$$\begin{aligned} & F_Y(x_1, \dots, x_n) \\ &= \begin{vmatrix} \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_1\|_q^q} |y_{1k}|^{q-1} \operatorname{sgn}(y_{1k}) y_{1k} & \cdots & \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_1\|_q^q} |y_{1k}|^{q-1} \operatorname{sgn}(y_{1k}) y_{1k} \\ \vdots & \ddots & \vdots \\ \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_n\|_q^q} |y_{nk}|^{q-1} \operatorname{sgn}(y_{nk}) y_{1k} & \cdots & \frac{\sqrt[n]{\|y_1, \dots, y_n\|_q}}{\sqrt[n]{n!} \|y_n\|_q^q} |y_{nk}|^{q-1} \operatorname{sgn}(y_{nk}) y_{nk} \end{vmatrix} \\ &= \frac{\|y_1, \dots, y_n\|_q}{n!} \begin{vmatrix} \sum_k \frac{1}{\|y_1\|_q^q} |y_{1k}|^{q-1} \operatorname{sgn}(y_{1k}) y_{1k} & \cdots & \sum_k \frac{1}{\|y_1\|_q^q} |y_{1k}|^{q-1} \operatorname{sgn}(y_{1k}) y_{1k} \\ \vdots & \ddots & \vdots \\ \sum_k \frac{1}{\|y_n\|_q^q} |y_{nk}|^{q-1} \operatorname{sgn}(y_{nk}) y_{1k} & \cdots & \sum_k \frac{1}{\|y_n\|_q^q} |y_{nk}|^{q-1} \operatorname{sgn}(y_{nk}) y_{nk} \end{vmatrix} \\ &= \frac{\|y_1, \dots, y_n\|_q}{n!} \begin{vmatrix} \frac{g(y_1, y_1)}{\|y_1\|_p^2} & \cdots & \frac{g(y_1, y_n)}{\|y_1\|_p^2} \\ \vdots & \ddots & \vdots \\ \frac{g(y_n, y_1)}{\|y_n\|_p^2} & \cdots & \frac{g(y_n, y_n)}{\|y_n\|_p^2} \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= \frac{\|y_1, \dots, y_n\|_q}{n!} \begin{vmatrix} \frac{g(y_1, y_1)}{\|y_1\|_p^2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{g(y_n, y_1)}{\|y_n\|_p^2} & \cdots & \cdots & \frac{g(y_n, y_n)}{\|y_n\|_p^2} \end{vmatrix} \\ &= \frac{\|y_1, \dots, y_n\|_q}{n!} \frac{g(y_1, y_1)}{\|y_1\|_p^2} \frac{g(y_2, y_2)}{\|y_2\|_p^2} \cdots \frac{g(y_n, y_n)}{\|y_n\|_p^2} \\ &= \frac{\|y_1, \dots, y_n\|_q}{n!}. \end{aligned}$$

Thus

$$\|F_Y\| \geq \frac{\|y_1, \dots, y_n\|_q}{n!}.$$

Hence

$$\frac{1}{n!} \|y_1, \dots, y_n\|_q \leq \|F_Y\| \leq \|y_1, \dots, y_n\|_q.$$

4. Concluding Remarks

In this paper, we have found a partial solution to this open problem given in (Batkunde et al. 2013) since we obtained $\|F_Y\| \equiv \|y_1, \dots, y_n\|_q$. But an exact solution still remains an open problem.

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