An invitation to Banach space geometry — Basic facts and some recent results

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The notions of strict and uniform convexity for Banach spaces as well as Clarkson's inequalities for L_p-spaces (J.A. Clarkson, Trans. AMS, 1936) are the origin of geometry of Banach spaces. In 1937 (Ann. of Math.) in connection with the famous result by Jordan and von Neumann (Ann. of Math., 1935) concerning characterization of inner product spaces, he introduced the *von Neumann-Jordan (NJ-)constant* $C_{NJ}(X)$ of a Banach space X and showed that $1 \leq C_{NJ}(X) \leq 2$ for all X and X is a Hilbert space if and only if $C_{NJ}(X) = 1$. He also calculated its exact value for L_p-spaces. The first systematic work on this constant was done by Kato-Takahashi (Proc. AMS, 1997), where it turned out that the C_{NJ}-constant is fairly informative.

When $C_{NJ}(X) < 2$? The answer to this natural question was given in Takahhashi-Kato (Nihonkai Math. J.,1998): $C_{NJ}(X) < 2$ if and only if X is *uniformly non-square (UNSQ).* The notion of UNSQ-ness was introduced by R.C. James in 1964 (Ann. of Math.). A quantative information of this property is described by the non-square constant $J(X)$ (Gao-Lau, Studia Math., 1991), which is now often called *James constant* as in Kato-Maligranda-Takahashi (Studia Math., 2001): $\sqrt{2} \leq J(X) \leq 2$ for alll X and X is UNSQ if and only if $J(X) < 2$. In 2006 a remarkable result appeared (García-Falset et al., J. Funct. Anal.): *A Banach space X has FPP if X is UNSQ.* Recently many geometric constants for Banach spaces have been investigated, in particular the inequality $C_{NJ}(X) \leq J(X)$ can be viewed as one of the most important results (Takahashi-Kato, JMAA, 2009; F.Wang-C.Yang, Proc. AMS, 2010; C.Yang-H.Li, Appl. Math. Lett., 2010).

In the sequence of our talks we shall present some basic facts and recent results on this subject. Direct sums of Banach spaces will be discussed, as well.