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CHARACTERIZATIONS FOR THE GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON MORREY SPACES

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Abstract. We present some characterizations for the boundedness of the generalized fractional integral operators on Morrey spaces. The characterizations follow from two key estimates, one for the norm of some functions in Morrey spaces, and another for the values of the corresponding fractional integrals. We prove three theorems about necessary and sufficient conditions. We show that these theorems are independent by giving some examples. We also obtain counterparts for the weak generalized Morrey spaces.

1. Introduction

In this paper, for a measurable function $\rho : (0,\infty) \to (0,\infty)$, we are interested in the generalized fractional integral operator I_ρ given by the formula

$$I_\rho f(x) := \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y)\,dy, \quad x \in \mathbb{R}^d,$$

for any suitable function f on $\mathbb{R}^d$. This generalized fractional integral operator was initilally investigated in [27]. Nowadays many authors have been culminating important observations about I_ρ especially in connection with Morrey spaces. These spaces cover Lebesgue spaces as special cases and seem to describe the behavior of I_ρ well. In order to highlight what we shall prove in this paper, we take up the works [3, 6, 7, 18, 22, 25, 28, 35], where we formulated sufficient conditions on ρ for I_ρ to be bounded on Morrey spaces $L_{p,\phi}$ with $1 \leqslant p < \infty$ and ϕ a function from $(0,\infty)$ to itself. We aim to show that these conditions are necessary as well. We characterize the boundedness by estimating the norm of the characteristic functions of balls and the function $\phi(|\cdot|)$, as well as the value of the corresponding fractional integrals.

Hereafter, we assume that

$$\int_0^1 \frac{\rho(s)}{s}\,ds < \infty,$$

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so that the fractional integrals $I_\rho f$ are well-defined, at least for characteristic functions of balls. In addition, we shall also assume that ρ satisfies the *growth condition*: there exist constants $C_1 > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r/2 < s \leqslant r} \rho(s) \leqslant C_1 \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s}\, ds, \quad r > 0. \tag{1.1}$$

This condition is weaker than the usual *doubling condition*: there exists a constant $C_2 > 0$ such that

$$\frac{1}{C_2} \leqslant \frac{\rho(r)}{\rho(s)} \leqslant C_2$$

whenever r and s satisfy

$$r, s > 0 \text{ and } \frac{1}{2} \leqslant \frac{r}{s} \leqslant 2.$$

See [40] for some examples and more explanation about these two conditions.

In the present paper we work on generalized Morrey spaces. For a certain function $\phi : (0,\infty) \to (0,\infty)$, we say that a function f belongs to the generalized Morrey space $L_{p,\phi} = L_{p,\phi}(\mathbb{R}^d)$, where $1 \leqslant p < \infty$, if

$$\|f : L_{p,\phi}\| := \sup_{a \in \mathbb{R}^d, r > 0} \frac{1}{\phi(r)} \left[\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx \right]^{1/p} < \infty.$$

Note that if $\phi(r) := r^{(\lambda - d)/p}$ for some $1 \leqslant p < \infty$ and $0 \leqslant \lambda < d$, then $L_{p,\phi}(\mathbb{R}^d) = L^{p,\lambda}(\mathbb{R}^d)$, see (1.8) below. In [26, p. 446] we justified that ϕ is a nonincreasing function such that $t \mapsto \phi(t)^p t^d$ is a nondecreasing for $L_{p,\phi}(\mathbb{R}^d) \neq \{0\}$. We refer to [25, 29, 32] and [40, Section 1] for more information about these spaces.

Here we shall assume that $\phi : (0,\infty) \to (0,\infty)$ is *almost decreasing* [that is, if $r \leqslant s$, then $\phi(r) \geqslant C_3 \phi(s)$], and that $r^d \phi^p(r)$ is *almost increasing*, [that is, if $r \leqslant s$, then $r^d \phi(r)^p \leqslant C_3' s^d \phi(s)^p$]. These two conditions implies that ϕ also satisfies the doubling condition. Denote by $\mathscr{G}_p$ the set of all functions $\phi : (0,\infty) \to (0,\infty)$ such that ϕ is almost decreasing and that $r \mapsto r^{d/p}\phi(r)$ is almost increasing. Now we present three different criteria for the boundedness of I_ρ. For covenience, write $\tilde{\rho}(r) := \int_0^r \frac{\rho(t)}{t}\, dt$. We prove the following theorems about the boundedness of I_ρ on generalized Morrey spaces.

THEOREM 1.1. *Let $1 < p < q < \infty$ and $\phi \in \mathscr{G}_p$. Assume*

$$\int_r^\infty \frac{\phi(s)\rho(s)}{s}\, ds \leqslant C\phi(r)\rho(r) \quad (r > 0) \tag{1.2}$$

for some constant $C > 0$. Then I_ρ is bounded from $L_{p,\phi}(\mathbb{R}^d)$ to $L_{q,\phi^{p/q}}(\mathbb{R}^d)$ if and only if there exists a constant $C' > 0$ such that

$$\tilde{\rho}(r) \leqslant C' \phi(r)^{p/q - 1} \quad (r > 0). \tag{1.3}$$

THEOREM 1.2. *Let* $1 < p < q < \infty$ *and let* $\phi \in \mathcal{G}_p$. *Assume*

$$\int_0^r \frac{\phi(s)s^{d/p}}{s}\,ds \leqslant C\phi(r)r^{d/p} \quad (r>0) \tag{1.4}$$

for some constant $C > 0$. *Then* I_ρ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q}}(\mathbb{R}^d)$ *if and only if there exists a constant* $C' > 0$ *such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C'\phi(r)^{p/q} \quad (r>0). \tag{1.5}$$

THEOREM 1.3. *Let* $\phi, \psi \in \mathcal{G}_1$. *Assume that*

$$\int_0^r \frac{\phi(s)s^{d}}{s}\,ds \leqslant C\phi(r)r^{d} \quad (r>0) \tag{1.6}$$

for some constant $C > 0$. *Then* I_ρ *is bounded from* $L_{1,\phi}(\mathbb{R}^d)$ *to* $L_{1,\psi}(\mathbb{R}^d)$ *if and only if there exists a constant* $C' > 0$ *such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C'\psi(r) \quad (r>0). \tag{1.7}$$

Note that the integral operators such as $(1-\Delta)^{-\alpha}$ and $L^{-\alpha}$, where L is a suitable elliptic differential operator and $\alpha > 0$, fall under this scope. Also, if a measurable function $V : \mathbb{R}^d \to (0,\infty)$ satisfies the reverse Hölder inequality, that is, there exist some constants $C > 0, q \gg 1$ such that, for all balls B, $\left(\int_B V(x)^q \frac{dx}{|B|}\right)^{1/q} \leqslant C\int_B V(x)\frac{dx}{|B|}$, then the operators $V^\gamma(-\Delta+V)^{-\beta}$ with $0 \leqslant \gamma \leqslant \beta \leqslant 1$ and $V^\gamma\partial_j(-\Delta+V)^{-\beta}$ with $0 \leqslant \gamma \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$, $\beta - \gamma \geqslant \frac{1}{2}$ and $j = 1,2,\cdots,d$ also fall under this scope [20]. We refer to [13, Sections 3 and 4] for a detailed description of these facts.

A few remarks concerning the conditions on the theorems may be in order.

REMARK 1.4.

(i) Theorems 1.1-1.3 extend those obtained in [7] where the authors considered the classical Riesz potential.

(ii) In Theorem 1.2, to prove the sufficiency, there is no need to assume (1.4).

(iii) The condition (1.5) appeared in [18] originally and it later appeared in a bilinear estimate of the form $g \cdot I_\alpha f$ (see [39, Theorem 1.6]).

(iv) In Theorem 1.3, to prove the sufficiency, there is no need to assume (1.6).

(v) It follows immediately that the right-hand side of (1.7) equals

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt = \int_0^\infty \frac{\phi(\max(r,t))\rho(t)}{t}\,dt.$$

(vi) The condition (1.7) is known to be sufficient in [28, Theorem 3.2].

Our results can be readily transplanted into those for Morrey spaces and Lebesgue spaces. For $1 \leqslant p < \infty$ and $0 \leqslant \lambda < d$, recall that the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^d)$ consists of all locally integrable functions f on $\mathbb{R}^d$ for which

$$\|f : L^{p,\lambda}\| := \sup_{a\in\mathbb{R}^d, r>0} \left[\frac{1}{r^\lambda}\int_{B(a,r)} |f(x)|^p dx\right]^{\frac{1}{p}} < \infty. \tag{1.8}$$

See [33] for more information about these spaces. Observe that, with norm coincidence, $L^{p,0}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for $1 \leqslant p < \infty$. We note that if $\rho(r) = r^\alpha$, with $0 < \alpha < d$, then $I_\rho = I_\alpha$ is the classical fractional integral operator, also known as the Riesz potential, which is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ *if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}$, where $1 < p, q < \infty$ [43]. The necessary part is usually proved by using the scaling arguments. See [6, 18, 19, 46] for some recent results on the boundedness properties of I_ρ.

Theorems 1.1–1.2 both characterize the kernel function ρ for which I_ρ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $1 < p < q < \infty$.

COROLLARY 1.5. *Let $1 < p < q < \infty$. The operator I_ρ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if $\rho(r) \leqslant C r^{d(1/p-1/q)}$ for all $r > 0$.*

For $\rho(r) = r^\alpha$, Corollary 1.5 reads that the operator I_ρ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if $\frac{\alpha}{d} = \frac{1}{p} - \frac{1}{q}$, where $1 < p < q < \infty$.

With Theorems 1.1–1.3 we can characterize the function ρ for which I_ρ is bounded from one Morrey space to another.

The next corollary generalizes the previous characterization in Corollary 1.5.

COROLLARY 1.6. *Let $1 < p < q < \infty$ and $0 \leqslant \lambda < d$. Assume that ρ satisfies (1.1). Then the operator I_ρ is bounded from $L^{p,\lambda}(\mathbb{R}^d)$ to $L^{q,\lambda}(\mathbb{R}^d)$ precisely when one of the following equivalent conditions holds.*

(a) $\rho(r) \leqslant C r^{(d-\lambda)(1/p-1/q)}$ *for all* $r > 0$.

(b) $\tilde{\rho}(r) = \int_0^r \frac{\rho(s)}{s} ds \leqslant C r^{(d-\lambda)(1/p-1/q)}$ *for all* $r > 0$.

In Sugano's modified setting, these conditions corresponds to [45, the formula (18)], for example.

Generalized Morrey spaces occur naturally. We give a proposition, which improves [41, Theorem 5.1]. We write

$$\ell_{-1,0}(r) := \begin{cases} \log \frac{1}{r} & (0 < r < e^{-1}), \\ 1 & (e^{-1} \leqslant r). \end{cases}$$

PROPOSITION 1.7. *Let $s \in (0,d)$. Define*

$$\psi(r) := \frac{r^d \ell_{-1,0}(r)}{(1+r)^s} \quad (r > 0).$$

Then there exists a constant $C_s > 0$ *such that*

$$\|f : L_{1,\psi}\| \leqslant C_s \|(1-\Delta)^{s/2} f : L^{1,d-s}\|$$

holds for all $f \in L^{1,d-s}(\mathbb{R}^d)$ *with* $(1-\Delta)^{s/2} f \in L^{1,d-s}(\mathbb{R}^d)$.

The proof will be given by Example 5 below. Note that we cannot delete $\ell_{-1,0}$ because it is a necessary condition for this estimate as Example 5 shows.

A passage of Theorems 1.1, 1.2 and 1.3 from generalized Morrey spaces to weak generalized Morrey spaces can be readily done. For $p \in [1,\infty)$ and $\phi : (0,\infty) \to (0,\infty)$, recall that $L_{p,\phi,\mathrm{weak}} = L_{p,\phi,\mathrm{weak}}(\mathbb{R}^d)$ is the set of all functions such that

$$\|f : L_{p,\phi,\mathrm{weak}}\| := \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\phi(r)} \left(\frac{\sup_{t>0} t^p m(B(a,r),f,t)}{|B(a,r)|} \right)^{1/p} < \infty, \tag{1.9}$$

where

$$m(B(a,r),f,t) := |\{x \in B(a,r) : |f(x)| > t\}|.$$

We remark that some prefer to use $M_{p,\phi}(\mathbb{R}^n)$ instead of $L_{p,\phi}(\mathbb{R}^n)$ and that, likewise, some prefer to use $WM_{p,\phi}(\mathbb{R}^n)$ instead of $L_{p,\phi,\mathrm{weak}}(\mathbb{R}^n)$.

We have counterparts of three theorems above.

THEOREM 1.8. *Let* $1 \leqslant p < q < \infty$ *and let* $\phi \in \mathcal{G}_p$. *Assume* (1.2). *Then* I_ρ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q},\mathrm{weak}}(\mathbb{R}^d)$ *if and only if* (1.3) *holds.*

THEOREM 1.9. *Let* $1 \leqslant p < q < \infty$ *and let* $\phi \in \mathcal{G}_p$. *Assume* (1.4). *Then* I_ρ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q},\mathrm{weak}}(\mathbb{R}^d)$ *if and only if* (1.5) *holds.*

THEOREM 1.10. *Let* $\phi, \psi \in \mathcal{G}_1$. *Assume that* (1.6). *Then* I_ρ *is bounded from* $L_{1,\phi}(\mathbb{R}^d)$ *to* $L_{1,\psi,\mathrm{weak}}(\mathbb{R}^d)$ *if and only if* (1.7) *holds.*

Here we will recall some works related to generalized fractional integral operators and generalized Morrey spaces.

In [17, 25, 31, 35], the authors generalized Morrey spaces to various directions. In [25], generalized Morrey spaces with variable growth condition are defined. The work [35] is a passage from [25] to the metric measure space whose underlying measure does not satisfy the doubling condition. The paper [34] is a counterpart to the weak type spaces. A further generalization is done in [17] and some related examples can be found in [21, 36]. As another generalization, in [31], Morrey spaces are generalized to martingale Morrey spaces. In [23], the authors applied generalized Morrey spaces to grasp the limiting case. In [33] Peetre proved there the boundedness of singular integral operator on generalized Morrey-Campanato spaces. The weak variants of the boundedness of the maximal operators, fractional integral operators and singular integral operators in generalized Morrey spaces were also investigated in the papers [11, 12].

In [2, 10, 14, 15, 24, 37, 42, 45] generalized fractional integral operators are investigated. The works [14, 15] are oriented to the boundedness of I_ρ and Guliyev and

Mustafayev used (1.2), (1.5) and (1.6) as a sufficient condition. In [42], the author placed themselves in the setting of non-doubling measure spaces and in [42, Theorem 2.1] the condition (1.6) showed up. In [24], the author extended the Gagliardo-Nirenberg inequality for I_α, the case when $\rho(t) = t^\alpha$, to the one for I_ρ. An intersection of two classical Morrey spaces with the same parameter p can be regarded as a generalized Morrey space. Sugano investigated the boundedness of fractional integral operators on intersections of Morrey spaces [45], where she postulated conditions stronger than (1.2) and (1.5).

Other operators such as commutators are taken up in [12, 37]. In [12, (5.1)] Guliyev, Karaman, Seymur and Shukurov essentially considered (1.5) in order to show the boundedness of the fractional maximal operator of order α.

In order to show the boundedness of commutators generated by BMO functions and I_α, in [37], the authors postulated ϕ on (1.2) and (1.5).

Hereafter, the letter C denotes a positive constant whose value may differ from line to line, which may depend on d, ρ, p and q, but not on the functions f and the variables x.

This paper is organized as follows: In Section 2, we shall give the norm estimates of the characteristic functions of balls and the function $\phi(|\cdot|)$, and calculate the image by I_ρ of these functions. Based upon these preliminary results, we shall prove Theorems 1.1–1.3 and 1.8–1.10, the main results in Section 3. Some examples are presented in Section 4 and they will show that Theorems 1.1–1.3 and 1.8–1.10 are independent.

2. Some norm and integral estimates

Let us first consider the characteristic functions of balls. For every $R > 0$, let $B_R := B(0,R)$ be the ball centered at 0 with radius R, and χ_{B_R} be the characteristic function of B_R. Recall $\tilde{\rho}(r) = \int_0^r \frac{\rho(s)}{s}\, ds$. Also we write $B(x,R) = \{y \in \mathbb{R}^d : |x-y| < R\}$.

The following lemmas will be used several times in this paper.

LEMMA 2.1. *There exists a constant $C > 0$ such that the inequality $\tilde{\rho}(R/2) \leqslant C I_\rho \chi_{B_R}(x)$ holds whenever $x \in B_{R/2}$ and $R > 0$.*

Proof. Take $x \in B_{R/2}$. We write the integral in full:

$$I_\rho \chi_{B_R}(x) = \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} \chi_{B_R}(y)\, dy = \int_{B_R} \frac{\rho(|x-y|)}{|x-y|^d}\, dy.$$

A geometric observation shows that $B(x,R/2) \subseteq B(0,R)$. Hence, we have

$$I_\rho \chi_{B_R}(x) \geqslant \int_{B(x,R/2)} \frac{\rho(|x-y|)}{|x-y|^d}\, dy = C \int_0^{R/2} \frac{\rho(s)}{s}\, ds.$$

Note that we only use the spherical coordinates to obtain the last integral. □

LEMMA 2.2. *For every $R > 0$ and a measurable function $\phi : (0,\infty) \to (0,\infty)$ satisfying the doubling condition*

$$\frac{1}{C} \leqslant \frac{\phi(s)}{\phi(r)} \leqslant C \quad (0 < r/2 \leqslant s \leqslant 2r), \tag{2.1}$$

the inequality

$$C^{-1} \int_{2R}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt \leqslant I_\rho g_R(x) \leqslant C \int_{2R/3}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt$$

holds whenever $x \in B_{R/3}$, where $g_R(x) := \phi(|x|)\chi_{B_R^c}(x)$.

Proof. We prove the right-hand inequality, the left-hand inequality being similar. A geometric observation shows that $|x-y| \sim |y|$ for all $x \in B_{R/3}$ and $y \in \mathbb{R}^d \setminus B_{2R/3}$ Since ϕ satisfies (2.1), then

$$\begin{aligned} I_\rho g_R(x) &= \int_{\mathbb{R}^d \setminus B_R} \frac{\phi(|y|)\rho(|x-y|)}{|x-y|^d}\,dy \\ &\leqslant \int_{\mathbb{R}^d \setminus B(x,2R/3)} \phi(|y|)\frac{\rho(|x-y|)}{|x-y|^d}\,dy \\ &= \int_{\mathbb{R}^d \setminus B_{2R/3}} \phi(|x-y|)\frac{\rho(|y|)}{|y|^d}\,dy \\ &\leqslant C\int_{2R/3}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt \text{ for } x \in B_{R/3}. \end{aligned}$$

It remains to write the most right-hand side in terms of the spherical coordinates. □

The lemma below gives an estimate for the norm of χ_{B_R} in $L_{p,\phi}(\mathbb{R}^d)$.

LEMMA 2.3. *Let $1 \leqslant p < \infty$ and $\phi \in \mathscr{G}_p$. There exists a constant $C > 0$ such that $C^{-1}\phi(R)^{-1} \leqslant \|\chi_{B_R} : L_{p,\phi}\| \leqslant C\phi(R)^{-1}$ for all $R > 0$.*

Lemma 2.3 is proven in [8, Proposition A] and [26, Lemma 3.3]. See [7] as well.

LEMMA 2.4. *Let $1 \leqslant p < \infty$ and $\phi \in \mathscr{G}_p$. Assume that there exists a constant $C > 0$ such that*

$$\int_0^R \phi(t)t^{d/p-1}\,dt \leqslant C\phi(R)R^{d/p} \quad (R > 0). \tag{2.2}$$

Then the function $x \mapsto \phi(|x|)$ belongs to $L_{p,\phi}(\mathbb{R}^d)$.

Proof. First note that (2.2) is equivalent to

$$\frac{1}{r^d}\int_0^r \phi(t)^p t^{d-1}\,dt \leqslant C\phi(r)^p \quad (r > 0) \tag{2.3}$$

in view of [30, Lemma 7.1]. Let $\phi_1(r) = \inf_{t \leqslant r} \phi(t)$. Then, $\phi_1(r)$ is a non-increasing function such that $L_{p,\phi}(\mathbb{R}^d)$ and $L_{p,\phi_1}(\mathbb{R}^d)$ are isomorphic [26, p. 446]. Hence, we can assume that ϕ itself is decreasing. In this case $x \mapsto \phi(|x|)$ is radial decreasing, so that

$$\left[\frac{1}{|B(a,r)|}\int_{B(a,r)} \phi(|x|)^p dx\right]^{1/p} \leqslant \left[\frac{1}{|B_r|}\int_{B_r} \phi(|x|)^p dx\right]^{1/p} \quad (a \in \mathbb{R}^d). \tag{2.4}$$

Combining (2.3) and (2.4) and using the spherical coordinate, we obtain the desired result. □

3. Proofs of main results

In this section we prove our main results, six theorems in Section 1. We prove their sufficiency in Subsection 3.1 and then prove their necessity in Subsection 3.2.

3.1. Proof of sufficiency

We remark that (1.5) includes (1.3). We prove the estimate (3.1). Once we prove (3.1), the estimate (3.1) gives us the boundedness of I_ρ from $L_{p,\phi}(\mathbb{R}^d)$ to $L_{q,\phi^{p/q}}(\mathbb{R}^d)$ as we shall see below. Here we use the fact that the Hardy-Littlewood maximal operator M is bounded on $L_{p,\phi}(\mathbb{R}^d)$, if $p > 1$ and ϕ is almost decreasing [29, 35]. See [25, 43, 44] for more information about M on the space $L_{p,\phi}(\mathbb{R}^d)$.

LEMMA 3.1. *Let $1 \leqslant p < q < \infty$ and let $\phi \in \mathscr{G}_p$. We assume (1.2) and (1.3), or we assume (1.5). If we normalize the norm of f by $\|f : L_{p,\phi}\| = 1$, then we have*

$$|I_\rho f(x)| \leqslant C\left([Mf(x)]^{p/q} + \inf_{r>0} \phi(r)^{p/q}\right), \quad x \in \mathbb{R}^d. \tag{3.1}$$

Proof. First we may assume that ϕ is continuous and strictly decreasing (see [29, Proposition 3.4]). Recall that k_1 and k_2 appeared in the condition (1.1) of ρ. Let $\rho^*(r) = \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} ds$. We have

$$|I_\rho f(x)| \leqslant C\left[\sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\rho^*(2^j r)}{(2^j r)^d} \int_{|x-y|<2^j r} |f(y)|dy\right]$$

for given $x \in \mathbb{R}^d$ and $r > 0$. Let Σ_I and Σ_{II} be the first and second summations above. Now we invoke the *overlapping property* in [40]:

$$\begin{aligned}
\sum_{j=-\infty}^{-1} \rho^*(2^j r) &\leqslant \sum_{j=-\infty}^{-1} \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)}{s} ds \\
&= \int_0^{k_2 r} \left(\sum_{j=-\infty}^{-1} \chi_{[2^j k_1 r, 2^j k_2 r]}(s)\right) \frac{\rho(s)}{s} ds \\
&\leqslant C \int_0^{k_2 r} \frac{\rho(s)}{s} ds \\
&\leqslant C\tilde{\rho}(k_2 r)
\end{aligned}$$

and

$$\sum_{j=0}^{\infty}\rho^*(2^j r)\phi(2^j r) = \int_{k_1 r}^{\infty}\left(\sum_{j=0}^{\infty}\chi_{[2^j k_1 r, 2^j k_2 r]}(s)\frac{\rho(s)}{s}\phi(2^j r)\right)ds$$
$$\leqslant C\int_{k_1 r}^{\infty}\left(\sum_{j=0}^{\infty}\chi_{[2^j k_1 r, 2^j k_2 r]}(s)\right)\frac{\rho(s)}{s}\phi(s)\,ds$$
$$\leqslant C\int_{k_1 r}^{\infty}\frac{\rho(s)}{s}\phi(s)\,ds.$$

Then, we have

$$\Sigma_I \leqslant C\sum_{j=-\infty}^{-1}\rho^*(2^j r)Mf(x) \leqslant C\tilde{\rho}(k_2 r)Mf(x) \leqslant C\phi(r)^{p/q-1}Mf(x),$$
$$\Sigma_{II} \leqslant C\sum_{j=0}^{\infty}\rho^*(2^j r)\phi(2^j r)\|f : L_{p,\phi}\| \leqslant C\int_{k_1 r}^{\infty}\frac{\rho(s)\phi(s)}{s}\,ds.$$

We use (1.2) or (1.5) now. By the doubling property of ϕ, we obtain $\Sigma_{II} \leqslant C\phi(r)^{p/q}$. Hence,

$$|I_\rho f(x)| \leqslant C\phi(r)^{p/q-1}\left[Mf(x)+\phi(r)\right] \quad (\text{for all } r>0). \tag{3.2}$$

First assume $Mf(x) \leqslant \inf_{r>0}\phi(r)$. Then, (3.1) is immediate from (3.2). Next, we assume $Mf(x) > \inf_{r>0}\phi(r)$. Since $\|f : L_{p,\phi}\| = 1$, we have

$$1 \geqslant \frac{1}{\phi(r)}\left(\frac{1}{B(x,r)}\int_{B(x,r)}|f(y)|^p\,dy\right)^{1/p} \geqslant \frac{1}{\phi(r)}\cdot\frac{1}{B(x,r)}\int_{B(x,r)}|f(y)|\,dy.$$

Hence

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \leqslant \phi(r)$$

for all $r>0$. This implies

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \leqslant \sup_{R>0}\phi(R)$$

for all $r>0$. Since $r>0$ and $x\in\mathbb{R}^d$ are arbitrary, it follows that $Mf(x) \leqslant \sup_{r>0}\phi(r)$. We can thus find $R>0$ such that $Mf(x) \sim 2\phi(R)$ and, with this R, we can obtain (3.1). □

Proof of Theorems 1.1 *and* 1.2 (Sufficiency). Let $\|f : L_{p,\phi}\| = 1$. Let $B = B(z,s)$ be an arbitrary ball. If we integrate (3.1), then we have

$$\frac{1}{|B|}\int_B |I_\rho f(x)|^q\,dx \leqslant C\left(\frac{1}{|B|}\int_B [Mf(x)]^p\,dx + \inf_{r>0}\phi(r)^p\right), \quad x\in\mathbb{R}^d.$$

If we devide both sides by $\phi(s)^p$, then we have

$$\frac{1}{\phi(s)^p|B|}\int_B |I_\rho f(x)|^q\,dx \leqslant C\left(\frac{1}{\phi(s)^p|B|}\int_B [Mf(x)]^p\,dx + 1\right) \leqslant C$$

by virtue of the boundedness of the maximal operator M on $L_{p,\phi}(\mathbb{R}^n)$. The ball B being arbitrary, we obtain the desired result. $\square$

Proof of Theorem 1.3 (Sufficiency). Let $\|f : L_{p,\phi}\| = 1$. For a ball $B(z,r)$, let $f_1 = f\chi_{B(z,2r)}$ and $f_2 = f - f_1$. Then a geometric observation shows $B(z,r) \subset B(y,3r)$ for all $y \in B(z,2r)$. Hence by the Fubini theorem and the normalization,

$$\begin{aligned}\int_{B(z,r)} |I_\rho f_1(x)|\,dx &\leqslant \int_{B(z,r)} \left(\int_{B(z,2r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\,dy\right) dx\\ &\leqslant \int_{B(z,2r)} \left(\int_{B(y,3r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\,dx\right) dy\\ &\leqslant C\tilde{\rho}(3r)\phi(3r)r^d\\ &\leqslant C\psi(r)r^d.\end{aligned}$$

Here for the last inequality we used (1.7) and the doubling condition of ψ. Thus, the estimate for f_1 is valid. As for f_2, we let $x \in B(z,r)$. Then we have

$$|I_\rho f_2(x)| \leqslant \int_{B(z,2r)^c} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\,dy \leqslant \int_{B(x,r)^c} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\,dy$$

and decomposing the right-hand side dyadically as we did in the proof of Theorem 1.1 for $\sum_{II}$, we obtain

$$|I_\rho f_2(x)| \leqslant \sum_{j=1}^{\infty} \int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\,dy \leqslant C\int_{2k_1 r}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt.$$

If we use (1.7) and the doubling condition of ψ, then we obtain $|I_\rho f_2(x)| \leqslant C\psi(r)$. Thus, the estimate for f_2 is valid as well. $\square$

Proof of Theorems 1.8 *and* 1.9 (Sufficiency). We normalize f so that we have $\|f : L_{p,\phi}\| = 1$. By virtue of (3.1), we have

$$\begin{aligned}&\frac{1}{\phi(r)^{p/q}}\left(\frac{t^q m(B(a,r),|I_\rho f|,t)}{|B(a,r)|}\right)^{1/q}\\ &\leqslant C\frac{1}{\phi(r)^{p/q}}\left(\frac{t^q m(B(a,r),[Mf]^{p/q},t/2)}{|B(a,r)|}\right)^{1/q}\\ &\quad + C\frac{1}{\phi(r)^{p/q}}\left(\frac{t^q m(B(a,r),\inf_{r'>0}\phi(r')^{p/q},t/2)}{|B(a,r)|}\right)^{1/q}\\ &\leqslant C\frac{1}{\phi(r)^{p/q}}\left(\frac{t^q m(B(a,r),[Mf]^{p/q},t/2)}{|B(a,r)|}\right)^{1/q} + C\frac{1}{\phi(r)^{p/q}}\inf_{r'>0}\phi(r')^{p/q}\\ &\leqslant C(\|Mf\|_{p,\phi,\mathrm{weak}})^{p/q} + C\\ &\leqslant C.\end{aligned}$$

Here for the second to the last inequality, we have used [29, Theorem 6.1]. Theorem 1.9 can be proved in the same way. □

Proof of Theorem 1.10 (Sufficiency). Under the condition (1.7), in the proof of the sufficiency of Theorem 1.3 we already established that I_ρ is bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi}(\mathbb{R}^d)$, which is stronger than the boundedness from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi,\text{weak}}(\mathbb{R}^d)$. □

REMARK 3.2. The proof of the sufficient part is similar to, but not the same as, that in [18, 28]. In this paper, we do not assume that ρ satisfies the doubling condition nor that ϕ is surjective, as we did in [18].

3.2. Proof of necessity

Proof of Theorem 1.1 (Necessity). Note that

$$\tilde{\rho}(R/2) \leqslant C\left[\frac{1}{|B_{R/2}|}\int_{B_{R/2}} |I_\rho \chi_{B_R}(x)|^q dx\right]^{1/q} \tag{3.3}$$

by virtue of a pointwise estimate in Lemma 2.1. Notice also that

$$\|I_\rho \chi_{B_R} : L_{q,\phi^{p/q}}\| \leqslant C\|\chi_{B_R} : L_{p,\phi}\| \tag{3.4}$$

since I_ρ is assumed bounded. If we combine (3.3), (3.4), Lemma 2.3, and the doubling property of ϕ, we have

$$\begin{aligned}
\tilde{\rho}(R/2) &\leqslant C\,\phi(R/2)^{p/q}\,\phi(R/2)^{-p/q}\left[\frac{1}{|B_{R/2}|}\int_{B_{R/2}} |I_\rho \chi_{B_R}(x)|^q dx\right]^{1/q}\\
&\leqslant C\,\phi(R/2)^{p/q}\,\|I_\rho \chi_{B_R} : L_{q,\phi^{p/q}}\|\\
&\leqslant C\,\phi(R/2)^{p/q}\,\phi(R)^{-1}\\
&\leqslant C\,\phi(R/2)^{p/q-1},
\end{aligned}$$

for all $R > 0$. This completes the proof. □

Proof of Theorem 1.2 (Necessity). The exactly same argument as we did for Theorem 1.1 works and we conclude

$$\int_0^r \frac{\rho(t)}{t}\,dt \leqslant C\phi(r)^{p/q-1}$$

for $r > 0$.

Meanwhile, by virtue of Lemma 2.2, we obtain

$$\int_{2R}^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \left(\frac{1}{R^d}\int_{B_{R/3}} I_\rho g(x)^q\,dx\right)^{1/q} \leqslant C\phi(R)^{p/q}\|I_\rho g_R : L_{q,\phi^{p/q}}\|.$$

Since I_ρ is bounded, we obtain

$$\int_{2R}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C\phi(R)^{p/q}\|g_R : L_{p,\phi}\|.$$

Now we invoke Lemma 2.4 to conclude

$$\int_{2R}^{\infty} \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C\phi(R)^{p/q} \leqslant C\phi(2R)^{p/q}.$$

Thus, Theorem 1.2 is proven. □

Proof of Theorem 1.3 (Necessity). Assume that I_ρ is bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi}(\mathbb{R}^d)$. By Lemma 2.1, we obtain

$$\tilde{\rho}(r) \sim r^{-d}\int_{B_{r/2}} I_\rho \chi_{B_r}(x)\,dx \leqslant \psi\left(\frac{r}{2}\right)\|I_\rho \chi_{B_r} : L_{1,\psi}\|.$$

Since $\psi \in \mathscr{G}_1$ and I_ρ is assumed bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi}(\mathbb{R}^d)$, it follows that $\tilde{\rho}(r) \leqslant C\psi(r)\|\chi_{B_r} : L_{1,\phi}\|$. Since $\|\chi_{B_r} : L_{1,\phi}\| \sim \phi(r)^{-1}$, we conclude $\tilde{\rho}(r) \leqslant C\frac{\psi(r)}{\phi(r)}$. Meanwhile, by Lemma 2.2, we have

$$\int_{r}^{\infty} \frac{\rho(t)\phi(t)}{t}\,dt \leqslant C\psi\left(\frac{r}{6}\right)\|I_\rho g_r : L_{1,\psi}\| \leqslant C\psi(r)\|g_r : L_{1,\phi}\| \leqslant C\psi(r).$$

Thus, Theorem 1.3 is proved. □

Proof of Theorem 1.8 (Necessity). Choose $C>0$ from Lemma 2.1 and write it for C_0. By Lemma 2.1 we have

$$\begin{aligned}\sup_{t>0} t^q m(B_{R/2}, I_\rho\chi_{B_R}(x), t) &\geqslant \left(\frac{\tilde{\rho}(R/2)}{C_0}\right)^q m\left(B_{R/2}, I_\rho\chi_{B_R}, \frac{\tilde{\rho}(R/2)}{C_0}\right)\\ &\geqslant \left(\frac{\tilde{\rho}(R/2)}{C_0}\right)^q |B_{R/2}|.\end{aligned}$$

If we arrange this inequality, then we have

$$\tilde{\rho}(R/2) \leqslant C\left[\frac{\sup_{t>0} t^q m(B_{R/2}, I_\rho\chi_{B_R}, t)}{|B_{R/2}|}\right]^{1/q}. \tag{3.5}$$

Therefore

$$\begin{aligned}\tilde{\rho}(R/2) &\leqslant C\,\phi(R/2)^{p/q}\,\phi(R/2)^{-p/q}\left[\frac{\sup_{t>0} t^q m(B_{R/2}, I_\rho\chi_{B_R}, t)}{|B_{R/2}|}\right]^{1/q}\\ &\leqslant C\,\phi(R/2)^{p/q}\,\|I_\rho\chi_{B_R} : L_{q,\phi^{p/q},\mathrm{weak}}\|\\ &\leqslant C\,\phi(R/2)^{p/q}\,\phi(R)^{-1}\\ &\leqslant C\,\phi(R/2)^{p/q-1},\end{aligned}$$

for all $R>0$. This completes the proof. □

Proof of Theorem 1.9 (Necessity). The exactly same argument as we did for Theorem 1.8 works and we conclude

$$\int_0^r \frac{\rho(t)}{t}\,dt \leqslant C\phi(r)^{p/q-1}$$

for $r>0$. Meanwhile, by virtue of Lemma 2.2, we obtain

$$\begin{aligned}
&\sup_{t>0} t^q m(B_{R/3}, I_\rho g_{B_R}, t)\\
&\geqslant \left(\frac{1}{C}\int_{2R}^{\infty}\frac{\phi(t)\rho(t)}{t}\,dt\right)^q m\left(B_{R/3}, I_\rho g_{B_R}, \frac{1}{C}\int_{2R}^{\infty}\frac{\phi(t)\rho(t)}{t}\,dt\right)\\
&\geqslant \left(\frac{1}{C}\int_{2R}^{\infty}\frac{\phi(t)\rho(t)}{t}\,dt\right)^q |B_{R/3}|.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\int_{2R}^{\infty}\frac{\phi(t)\rho(t)}{t}\,dt &\leqslant C\left[\frac{\sup_{t>0} t^q m(B_{R/3}, I_\rho g_{B_R}, t)}{|B_{R/3}|}\right]^{1/q}\\
&\leqslant C\phi(R/3)^{p/q}\|I_\rho g_R : L_{q,\phi^{p/q},\mathrm{weak}}\|\\
&\leqslant C\phi(R/3)^{p/q}\|g_R : L_{p,\phi}\| \leqslant C\phi(R/3)^{p/q} \leqslant C\phi(2R)^{p/q}.
\end{aligned}$$

Thus, Theorem 1.9 is proven. □

Proof of Theorem 1.10 (Necessity). Assume that I_ρ is bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi,\mathrm{weak}}(\mathbb{R}^d)$. By Lemma 2.1, we obtain

$$\tilde{\rho}(R/2) \leqslant C\frac{\sup_{t>0} t m(B_{R/2}, I_\rho \chi_{B_R}(x), t)}{|B_{R/2}|} \leqslant \psi\left(\frac{R}{2}\right)\|I_\rho \chi_{B_R} : L_{1,\psi,\mathrm{weak}}\|.$$

Since $\psi \in \mathscr{G}_1$ and I_ρ is assumed bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi,\mathrm{weak}}(\mathbb{R}^d)$, it follows that

$$\tilde{\rho}(r) \leqslant C\psi(r)\|\chi_{B_{2r}} : L_{1,\phi}\| \sim \frac{\psi(r)}{\phi(r)}.$$

Meanwhile, by Lemma 2.2, we have

$$\begin{aligned}
\int_r^{\infty}\frac{\rho(t)\phi(t)}{t}\,dt &\leqslant C\psi\left(\frac{r}{6}\right)\|I_\rho g_r : L_{1,\psi,\mathrm{weak}}\|\\
&\leqslant C\psi(r)\|g_r : L_{1,\phi}\|\\
&\leqslant C\psi(r).
\end{aligned}$$

Thus, Theorem 1.10 is proved. □

4. Examples showing Theorems 1.1–1.3 are independent

In this section, we show by examples that Theorems 1.1–1.3 are of independent interest. Here and below we write

$$\ell_{\beta_1,\beta_2}(r) := \begin{cases} \left(\log \frac{1}{r}\right)^{-\beta_1} & (0 < r < e^{-1}), \\ 1 & (e^{-1} \leqslant r \leqslant e), \\ (\log r)^{\beta_2} & (e < r). \end{cases}$$

This function is used to describe the "log"-growth and "log"-decay properties. Also, we fix p and q so that $1 < p < q < \infty$.

EXAMPLE 1. Let μ_1, μ_2 satisfy $\mu_1, \mu_2 \geqslant 0$. Set $\alpha := \frac{d}{p} - \frac{d}{q}$ and $\beta_i := \left(\frac{p}{q} - 1\right)\mu_i$ for $i = 1, 2$. Define $\rho(r) := r^\alpha \ell_{\beta_1,\beta_2}(r)$, $\quad \phi(r) := r^{-\frac{d}{p}} \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair (ρ, ϕ) fulfills the assumptions (1.2) and (1.3) in Theorem 1.1 but fails (1.4) in Theorem 1.2. More precisely, since $\alpha > 0$, we have $\tilde{\rho}(r) \sim \rho(r)$ and $\int_r^\infty \frac{\phi(t)\rho(t)}{t} dt \sim \phi(r)\rho(r)$.

Example 1 is an endpoint case of the next example.

EXAMPLE 2. Let λ satisfy $0 < \left(\frac{p}{q} - 1\right)\lambda < d$ and $-\frac{d}{p} < \lambda < 0$. Take μ_1, μ_2 arbitrarily. Set $\alpha := \left(\frac{p}{q} - 1\right)\lambda$ and $\beta_i := \left(\frac{p}{q} - 1\right)\mu_i$ for $i = 1, 2$. Define $\rho(r) := r^\alpha \ell_{\beta_1,\beta_2}(r)$ and $\phi(r) := r^\lambda \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair (ρ, ϕ) fulfills the assumptions (1.2)–(1.4) in Theorems 1.1 and 1.2. Indeed, $\tilde{\rho}(r) \sim \rho(r)$, $\quad \int_r^\infty \frac{\phi(t)\rho(t)}{t} dt \sim \phi(r)\rho(r)$.

The next example concerns the case when the spaces are close to $L^\infty(\mathbb{R}^d)$ and the smoothing order of I_ρ is "almost 0".

EXAMPLE 3. Let $\mu_1, \mu_2 < 0$. Set $\beta_1 := \left(\frac{p}{q} - 1\right)\mu_1 + 1 \in (1, \infty)$ and $\beta_2 := \left(\frac{p}{q} - 1\right)\mu_2 - 1 \in (-1, \infty)$. Define $\rho(r) := \ell_{\beta_1,\beta_2}(r)$ and $\phi(r) := \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair (ρ, ϕ) fulfills the assumptions (1.4) and (1.5) in Theorem 1.2 but fails (1.2) in Theorem 1.1. More precisely, for all $r > 0$, we have $\tilde{\rho}(r) \sim \ell_{\beta_1 - 1, \beta_2 + 1}(r)$, and $\int_r^\infty \frac{\phi(t)\rho(t)}{t} dt \sim \ell_{\mu_1+\beta_1-1, \mu_2+\beta_2+1}(r)$.

We consider a case when the target space is close to $L^\infty(\mathbb{R}^d)$.

EXAMPLE 4. Let α, β_1, μ_1, μ_2 satisfy $0 < \alpha < \frac{d}{p}$, $\mu_1 + \beta_1 < 1$, $\mu_2 < 0$. Set $\beta_2 := \left(\frac{p}{q} - 1\right)\mu_2 - 1 \in (-1, \infty)$. Define $\rho(r) := \min(1, r^\alpha)\ell_{\beta_1,\beta_2}(r)$ and $\phi(r) := \max(1, r^{-\alpha})\ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair (ρ, ϕ) fulfills the assumptions (1.4) and (1.5) in Theorem 1.2 but fails (1.2) in Theorem 1.1. More precisely, $\phi(r)\tilde{\rho}(r) \sim \ell_{\mu_1+\beta_1, \mu_2+\beta_2+1}(r)$ and $\int_r^\infty \frac{\phi(t)\rho(t)}{t} dt \sim \ell_{\mu_1+\beta_1-1, \mu_2+\beta_2+1}(r)$ for $r > 0$.

We conclude this paper by proving Proposition 1.7.

EXAMPLE 5. Let $0 < s < d$. Define $\phi(r) := r^{-s}$ and $\psi(r) := (1+r)^{-s}\ell_{-1,0}(r)$ for $r > 0$. Let $\rho(r) := r^d G_s(r)$, where G_s denotes the Bessel kernel, the kernel of $(1-\Delta)^{s/2}$. Observe that $\tilde{\rho}(r) \sim \min(r^s, 1)$ and hence $\phi(r)\tilde{\rho}(r) \sim \min(1, r^{-s})$. Note also that $\int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \begin{cases} \log(e/r) & (r < 1), \\ r^{d-s}G_s(r) & (r \geqslant 1). \end{cases}$ Then we have $\phi(r)\tilde{\rho}(r) + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \psi(r)$. Hence it follows from Theorem 1.3 that $\|I_\rho f : L_{1,\psi}\| \leqslant C\|f : L_{1,\phi}\|$, extending Proposition 1.7. This triple (ρ, ϕ, ψ) fulfills the assumptions (1.7) and (1.6) in Thoerem 1.3 but it fails (1.2) in Theorem 1.1 and (1.5) in Theorem 1.2.

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