Hardy and Heisenberg's Inequality in Morrey Spaces

H. Gunawan*

*http://personal.fmipa.itb.ac.id/hgunawan/

Analysis and Geometry Group
Bandung Institute of Technology
Bandung 40132, INDONESIA

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Abstract

In this talk we shall discuss Hardy-type and Heisenberg-type inequalities in Morrey spaces. The Hardy-type inequality is proved via an Olsen's inequality in Morrey spaces. We then use the Hardy-type inequality together with an interpolation inequality for the fractional power of the Laplacian to obtain the Heisenberg-type inequality in Morrey spaces. The scenario is similar to that of Ciatti, Cowling, and Ricci (2015).

Introduction

One of important results in Fourier Analysis is **Heisenberg's Uncertainty Principle**, which states that a signal cannot have bounded frequency in a bounded time. If $f : \mathbb{R} \to \mathbb{C}$ and $\widehat{f}$ denotes the Fourier transform of f, then f and $\widehat{f}$ cannot be both supported on a finite interval except for $f \equiv 0$.
More precisely, if $f \in L^2(\mathbb{R})$, then

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) \geq \frac{1}{4}$$

with $\Delta_a f := \frac{\int_{\mathbb{R}}(x-a)^2|f(x)|^2 dx}{\int_{\mathbb{R}}|f(x)|^2 dx}$ being the **dispersion** of f around a.
In this talk, a generalization of this inequality in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and in Morrey spaces will be discussed.

Definition

For $f \in L^1(\mathbb{R})$, the **Fourier transform** of f, denoted by $\widehat{f}$, is defined by

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx, \quad \xi \in \mathbb{R}.$$

Using the fact that $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$ is *dense* in $L^2(\mathbb{R})$, the definition of Fourier transforms can be extended to $L^2(\mathbb{R})$, and finally to $L^p(\mathbb{R})$ for $1 \leq p \leq 2$.

Some Properties of Fourier Transforms

The Fourier transform $\mathcal{F}$ maps a function f in $L^1(\mathbb{R})$ to the function $\mathcal{F}f := \widehat{f}$ in $C_0(\mathbb{R})$.

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`Note.`

• $C_0(\mathbb{R})$ is the space of continuous and bounded functions on $\mathbb{R}$ vanishing at $\pm\infty$.

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Note.

• $C_0(\mathbb{R})$ is the space of continuous and bounded functions on $\mathbb{R}$ vanishing at $\pm\infty$.

• The notation $\mathcal{F}f$ will be used interchangeably with the notation $\widehat{f}$ for the Fourier transforms of f.

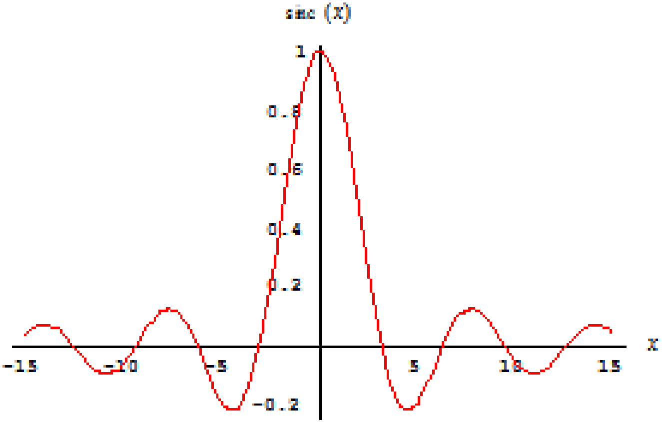
Examples

Example 1. If $f = \chi_{[-a,a]}$ $(a > 0)$, then $\widehat{f}(\xi) = 2\frac{\sin a\xi}{\xi}$.

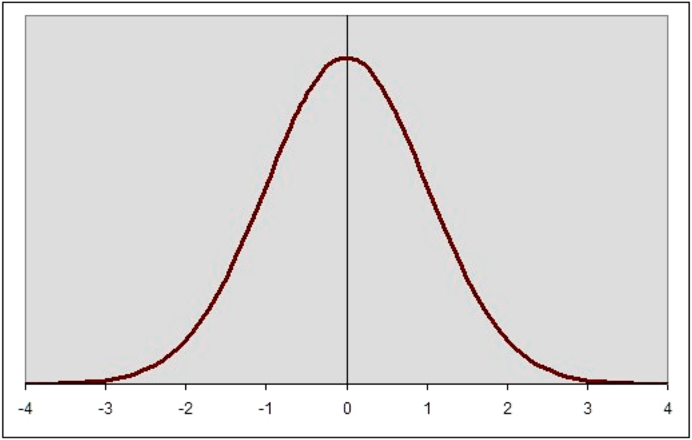
Example 2. If $f(x) = \frac{1}{x^2+a^2}$ $(a > 0)$, then $\widehat{f}(\xi) = \frac{\pi}{a}e^{-a|\xi|}$.

Example 3. If $f(x) = e^{-ax^2}$ $(a > 0)$, then $\widehat{f}(\xi) = \sqrt{\frac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$.

The Sinc Function



The Gauss Function



Fourier Transforms and Derivatives

If f is continuous and piecewise smooth, and $f' \in L^1(\mathbb{R})$, then

$$[f']\widehat{\ }(\xi) = i\xi \widehat{f}(\xi).$$

Conversely, if $xf(x) \in L^1(\mathbb{R})$, then

$$\mathcal{F}[xf(x)] = i[\widehat{f}]'(\xi).$$

Plancherel's Identity

Theorem (Plancherel's Identity)

If $f \in L^2(\mathbb{R})$, *then* $\|\widehat{f}\|_2 = \sqrt{2\pi}\,\|f\|_2$.

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Note.

• Plancherel's identity states that the Fourier transforms is an **isometry** (*modulo constant*) on $L^2(\mathbb{R})$.

• In other words, the Fourier transforms 'preserves energy' on $L^2(\mathbb{R})$.

• $\|f\|_2 := \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2}$ is a **norm** on $L^2(\mathbb{R})$.

Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle was first stated in 1927 by **Werner Heisenberg** (1901-1976), and proved mathematically by **W. Pauli** and **H. Weyl** in 1928.

Heisenberg is a physicist, a pioneer in quantum mechanics.

Heisenberg's Uncertainty Principle states that the more precise the position of a particle, the more imprecise its momentum; and vice versa.

Werner Heisenberg (1901-1976)

Heisenberg's Inequality

If $f(x) \in L^2(\mathbb{R})$, then

$$(\Delta_a f)(\Delta_\alpha \widehat{f}) \geq \frac{1}{4},$$

for every $a, \alpha \in \mathbb{R}$.

Here,

$$\Delta_a f := \frac{\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 dx}{\int_{\mathbb{R}} |f(x)|^2 dx}$$

representes the **dispersion** of f around a.

Interpretation

$\Delta_a f$ measures how much f is distributed away from a.

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Heisenberg's Uncertainty Principle states that $\Delta_a f$ and $\Delta_\alpha \widehat{f}$ cannot be both small. If one is less than $\frac{1}{2}$, then the other should be greater than $\frac{1}{2}$.

The lesser $\Delta_a f$, the greater $\Delta_\alpha f$; and vice versa.

Heisenberg's Inequality: The case $a = 0$ and $\alpha = 0$

If $f \in L^2(\mathbb{R})$, and

$$\left(\int_{\mathbb{R}} |xf(x)|^2 dx\right)\left(\int_{\mathbb{R}} |\xi\widehat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)\left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi\right).$$

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Equivalently,

$$\Big(\int_{\mathbb{R}} |xf(x)|^2 dx\Big)\Big(\int_{\mathbb{R}} |f'(x)|^2 dx\Big) \geq \frac{1}{4}\Big(\int_{\mathbb{R}} |f(x)|^2 dx\Big)^2.$$

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Equivalently,

$$\left(\int_{\mathbb{R}} |xf(x)|^2 dx\right)\left(\int_{\mathbb{R}} |f'(x)|^2 dx\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^2.$$

If we can prove this inequality, then by substituting $F(x) := e^{-i\alpha x} f(x+a)$ we get Heisenberg's inequality $(\Delta_a f)(\Delta_\alpha \widehat{f}) = (\Delta_0 F)(\Delta_0 \widehat{f}) \geq \frac{1}{4}$.

Proof of Heisenberg's Inequality

If $xf(x) \notin L^2(\mathbb{R}$ or $f'(x) \notin L^2(\mathbb{R})$, then the left hand side is infinite, so that the inequality holds. If $xf(x)$ and $f'(x) \in L^2(\mathbb{R})$, then by **integration by parts** we get

$$\int_{\mathbb{R}} |f(x)|^2 dx = -2\,\mathrm{Re} \int_{\mathbb{R}} \overline{xf(x)} f'(x)\, dx.$$

By using **Cauchy-Schwarz inequality**, we obtain

$$\int_{\mathbb{R}} |f(x)|^2 dx \leq 2\Big(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\Big)^{1/2} \Big(\int_{\mathbb{R}} |f'(x)|^2 dx\Big)^{1/2}.$$

Squaring both sides, we arrive at Heisenberg's inequality. □

When Does the Inequality Become an Equality?

Heisenberg's Inequality becomes an equality if and only if

$$f(x) = Ce^{-kx^2/2},$$

that is, f is a Gauss function.

Heisenberg's Inequality in $L^2(\mathbb{R}^n)$

Heisenberg's inequality which we have discussed is an inequality in $L^2(\mathbb{R})$.

A generalization of Heisenberg's inequality in $L^2(\mathbb{R}^n)$ may be formulated as:

$$\left(\int_{\mathbb{R}^n} |f(x)|^2\,dx\right)^2 \leq C(n,\alpha)\left(\int_{\mathbb{R}^n} \big||x|^\alpha f(x)\big|^2 dx\right)\left(\int_{\mathbb{R}^n} \big|D^{\alpha/2} f(x)\big|^2 dx\right)$$

where $D^{\alpha/2} = (-\Delta)^{\alpha/2}$ is defined via

$$[D^{\alpha/2} f]\widehat{\ }(\xi) = |\xi|^\alpha \widehat{f}(\xi)$$

for $0 < \alpha < n$.[1]

[1] W. Beckner, Pitt's inequality and the uncertainty principle, Proc. Amer. Math. Soc. **123** (1995)

Heisenberg's Inequality in the Norm Notation

In the norm notation, Heisenberg's inequality in $L^2(\mathbb{R}^n)$ may be rewritten as

$$\begin{aligned} \|f\|_2^4 &\leq C(n,\alpha)\ \||\cdot|^\alpha f\|_2^2\ \||\cdot|^\alpha \widehat{f}\|_2^2 \\ &\leq C(n,\alpha)\ \||\cdot|^\alpha f\|_2^2\ \|D^{\alpha/2} f\|_2^2 \end{aligned}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\mathbb{R}^n)$.

The Boundedness of T_α on $L^p(\mathbb{R}^n)$

To extend Heisenberg's inequality to $L^p(\mathbb{R}^n)$, consider the operator

$$T_\alpha : f \mapsto |\cdot|^{-\alpha} D^{-\alpha/2} f = T_\alpha f, \quad f \in C_c^\infty(\mathbb{R}^n).$$

[2] W. Beckner, Pitt's inequality with sharp convolution estimates, Proc. Amer. Math. Soc. **136** (2008)

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$$T_\alpha : f \mapsto |\cdot|^{-\alpha} D^{-\alpha/2} f = T_\alpha f, \quad f \in C_c^\infty(\mathbb{R}^n).$$

Here $I_\alpha := D^{-\alpha/2}$ is nothing but the **fractional integral operator**, which is a convolution operator with kernel $C_\alpha |\cdot|^{\alpha-n}$.
According to Beckner[2], T_α is bounded on $L^p(\mathbb{R}^n)$, that is,

$$\|T_\alpha f\|_p \le |||T_\alpha|||_{p,p} \|f\|_p$$

where $|||T_\alpha|||_{p,p}$ denotes the norm of the operator T_α from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

[2] W. Beckner, Pitt's inequality with sharp convolution estimates, Proc. Amer. Math. Soc. **136** (2008)

Hardy-type Inequality in $L^p(\mathbb{R}^n)$

As a consequence of the boundedness of T_α on $L^p(\mathbb{R}^n)$, we obtain an inequality for $|\cdot|^{-\alpha} f$.

To be precise, if we apply T_α on $D^{\alpha/2} f$, then we obtain

$$\| |\cdot|^{-\alpha} f \|_p = \| T_\alpha (D^{\alpha/2} f) \|_p \leq \| | T_\alpha | \|_{p,p} \| D^{\alpha/2} f \|_p$$

for $f \in C_c^\infty(\mathbb{R}^n)$.

The inequality may be viewed as a **Hardy-type inequality** in L^p spaces.

An Extension of Heisenberg's Inequality in $L^p(\mathbb{R}^n)$

As a consequence, we obtain an extension of Heisenberg's inequality in $L^p(\mathbb{R}^n)$:

$$\|f\|_p \leq C \| |\cdot|^\beta f\|_q^{\gamma/(\beta+\gamma)} \|D^{\gamma/2} f\|_r^{\beta/(\beta+\gamma)}$$

where $\beta > 0$, $0 < \gamma < \frac{n}{r}$, $p > 1$, $q \geq 1$, $r > 1$, and $\frac{\beta+\gamma}{p} = \frac{\gamma}{q} + \frac{\beta}{r}$.

Proof

Write

$$f(x) = \left[|x|^\beta f(x)\right]^{\gamma/(\beta+\gamma)} \left[|x|^{-\gamma} f(x)\right]^{\beta/(\beta+\gamma)}.$$

By Hölder's inequality, we obtain

$$\begin{aligned} \|f\|_p &\leq \||\cdot|^\beta f\|_q^{\gamma/(\beta+\gamma)} \||\cdot|^{-\gamma} f\|_r^{\beta/(\beta+\gamma)} \\ &\leq C\||\cdot|^\beta f\|_q^{\gamma/(\beta+\gamma)} \|D^{\gamma/2} f\|_r^{\beta/(\beta+\gamma)} \end{aligned}$$

as desired. □

Ciatti-Cowling-Ricci's Result

In 2015, P. Ciatti, M. Cowling, and F. Ricci[3], proved a Heisenberg's inequality on stratified Lie groups.

In particular, using an interpolation inequality for $D^{\delta/2}$, they obtained

$$\begin{aligned} \|f\|_p &\le \| |\cdot|^\beta f\|_q^{\delta/(\beta+\delta)} \| |\cdot|^{-\delta} f\|_r^{\beta/(\beta+\delta)} \\ &\le C \| |\cdot|^\beta f\|_q^{\delta/(\beta+\delta)} \|D^{\delta/2} f\|_r^{\beta/(\beta+\delta)} \end{aligned}$$

for $\beta, \delta > 0$, $p > 1$, $q \ge 1$, $r > 1$, and $\frac{\beta+\delta}{p} = \frac{\delta}{q} + \frac{\beta}{r}$.

If in the previous inequality there is a restriction for the value of γ, here there is no restriction for the value of δ — which means that Heisenberg's inequality for f may invlove the 'derivative' of any order of f.

[3] P. Ciatti, M. Cowling, and F. Ricci, Hardy and uncertainty inequalities on stratified Lie groups, Adv. Math. **277** (2015)

My Co-authors

In 2017, I met my co-authors (D. Hakim, E. Nakai, dan Y. Sawano) in Japan and we succeeded in proving Heisenberg's inequality in Morrey spaces, which I will explain next.

Morrey Spaces

For $1 \leq p \leq q < \infty$, we define the *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^n)$ by

$$\mathcal{M}^p_q := \{f \in L^p_{\text{loc}}(\mathbb{R}^n) \,:\, \|f\|_{\mathcal{M}^p_q} < \infty\},$$

where $\|\cdot\|_{\mathcal{M}^p_q}$ is given by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n,\ r>0} |B(a,r)|^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Here, $B(a,r)$ denotes the open ball in $\mathbb{R}^n$ centered at a with radius r, and $|B(a,r)| = c_n r^n$ is its Lebesgue measure.

Morrey spaces were first introduced by C.B. Morrey in 1938.[4]

[4] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938)

Recall the **fractional integral operator** I_α, which is defined for $0 < \alpha < n$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

for any locally integrable function f on $\mathbb{R}^n$.

Note that $I_\alpha \sim (-\Delta)^{-\alpha/2}$.

The following theorem states that I_α is bounded from $\mathcal{M}^p_q$ to $\mathcal{M}^s_t$ (see the works of Adams[5] and Chiarenza & Frasca[6] for the proof):

Theorem 5.1

Let $1 < p \le q < \frac{n}{\alpha}$ and $1 < s \le t < \infty$. If

$$\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n} \text{ and } \frac{p}{q} = \frac{s}{t},$$

then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q},$$

for every $f \in \mathcal{M}^p_q$.

[5] D.R. Adams, A note on Riesz potentials, Duke Math. J. **42** (1975)

[6] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. **7** (1987)

A Hardy-type Inequality in Morrey Spaces

As in L^p spaces, we also have the following Hardy-type inequality in Morrey spaces:

Theorem 5.2

Let $1 < p \leq q < \infty$ *and* $0 < \alpha < \frac{n}{q}$. *Then*

$$\| |\cdot|^{-\alpha} g \|_{\mathcal{M}_q^p} \lesssim \| (-\Delta)^{\frac{\alpha}{2}} g \|_{\mathcal{M}_q^p}$$

for every $g \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$.

Proof

Case 1: $1 < p < q < \infty$

Let $u := \frac{np}{\alpha q} < \frac{n}{\alpha} =: v$. Then $W(x) := |x|^{-\alpha} \in \mathcal{M}_v^u(\mathbb{R}^n)$. Let $f = (-\Delta)^{\frac{\alpha}{2}} g \in \mathcal{M}_q^p(\mathbb{R}^n)$. By Olsen's inequality, we obtain

$$\||\cdot|^{-\alpha} g\|_{\mathcal{M}_q^p} = \|W \cdot I_\alpha f\|_{\mathcal{M}_q^p} \lesssim \|W\|_{\mathcal{M}_v^u} \|(-\Delta)^{\frac{\alpha}{2}} g\|_{\mathcal{M}_q^p}$$

for every $g \in C_c^\infty(\mathbb{R}^n)$.

Proof

Case 1: $1 < p < q < \infty$
Let $u := \frac{np}{\alpha q} < \frac{n}{\alpha} =: v$. Then $W(x) := |x|^{-\alpha} \in \mathcal{M}^u_v(\mathbb{R}^n)$. Let $f = (-\Delta)^{\frac{\alpha}{2}} g \in \mathcal{M}^p_q(\mathbb{R}^n)$. By Olsen's inequality, we obtain

$$\||\cdot|^{-\alpha} g\|_{\mathcal{M}^p_q} = \|W \cdot I_\alpha f\|_{\mathcal{M}^p_q} \lesssim \|W\|_{\mathcal{M}^u_v} \|(-\Delta)^{\frac{\alpha}{2}} g\|_{\mathcal{M}^p_q}$$

for every $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

Case 2: $1 < p = q < \frac{n}{\alpha}$
For $1 \le q < \frac{n}{\alpha}$, we have (for every $g \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$)

$$\||\cdot|^{-\alpha} g\|_{wL^q} \lesssim \|W\|_{wL^v} \|(-\Delta)^{-\frac{\alpha}{2}} f\|_{wL^t} \lesssim \|W\|_{wL^v} \|(-\Delta)^{\frac{\alpha}{2}} g\|_{L^q},$$

with $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}$ and $v = \frac{n}{\alpha}$. By Marcinkiewicz interpolation, we obtain the strong type inequality for $1 < q < \frac{n}{\alpha}$.

Heisenberg's Inequality in Morrey Spaces, 1st Version

As a corollary of the Hardy-type inequality, we have:

Theorem 5.3

Let $1 < p \le q < \infty,\ 1 \le p_2 \le q_2 < \infty,\ \beta > 0$, *and* $0 < \gamma < \frac{n}{q}$. *If* $\frac{\beta+\gamma}{p_0} = \frac{\beta}{p} + \frac{\gamma}{p_2}$ *and* $\frac{\beta+\gamma}{q_0} = \frac{\beta}{q} + \frac{\gamma}{q_2}$, *then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\gamma/2} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^{p}_{q}}$$

for every $g \in C_c^\infty(\mathbb{R}^n)$.

An Interpolation Inequality for $(-\Delta)^{\alpha/2}f$

Using the boundedness of the imaginary power of the Laplacian, $(-\Delta)^{iu}$, on Morrey spaces[7], we obtain the following interpolation inequality for $(-\Delta)^{\alpha/2}f$:

Theorem 5.4

For $\alpha \geq 0$ and $0 \leq \theta \leq 1$, we have

$$\|(-\Delta)^{\alpha\theta/2}f\|_{\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}}\|(-\Delta)^{\alpha/2}f\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, with $1 < p_0 \leq q_0 < \infty$ and $1 < p_1 \leq q_1 < \infty$.

[7] G, D. Hakim, E. Nakai, Y. Sawano, The Hardy and Heisenberg inequalities in Morrey spaces, Bull. Aust. Math. Soc. **97** (2018)

Heisenberg's Inequality in Morrey Spaces, 2nd Version

Using the interpolation inequality for $(-\Delta)^{\alpha/2} f$, we obtain:

Theorem 5.5

Let $1 < p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, $\beta > 0$, *and* $\beta, \delta > 0$. *If* $\frac{\beta+\delta}{p_0} = \frac{\beta}{p_1} + \frac{\delta}{p_2}$ *and* $\frac{\beta+\delta}{q_0} = \frac{\beta}{q_1} + \frac{\delta}{q_2}$, *then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \| |\cdot|^{\beta} g\|^{\delta/(\beta+\delta)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta/(\beta+\delta)}_{\mathcal{M}^{p_1}_{q_1}}$$

for every $g \in C^{\infty}_{c}(\mathbb{R}^n)$.

Note: Unlike γ, here δ can be as large as possible.

Proof

If $\delta < \frac{n}{q_1}$, we do not have to do anything – the inequality is the same as in Theorem 5.3. Otherwise, we set $\gamma = \delta\theta$ and apply the interpolation inequality

$$\|(-\Delta)^{\delta\theta/2} g\|_{\mathcal{M}^p_q} \lesssim \|g\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|(-\Delta)^{\delta/2} g\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}}$$

for $0 < \theta < \frac{n}{\delta q_1}$, so that the inequality in Theorem 5.3 becomes

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta\theta/(\beta+\gamma)}_{\mathcal{M}^{p_1}_{q_1}} \|g\|^{\beta(1-\theta)/(\beta+\gamma)}_{\mathcal{M}^{p_0}_{q_0}}.$$

Rearranging the expression, we get the desired inequality.

Most results presented here are extracted from the paper by G., Hakim, Limanta, and Masta, "Hardy and Heisenberg inequalities in Morrey spaces", Bull. Aust. Math. Soc. **97** (2018).

I personally would like to thank Prof. Daher for inviting me to Morocco and giving me the chance to present this talk at 12th WHAIG.

THANK YOU FOR YOUR KIND ATTENTION.