Journal of Mathematical Analysis
ISSN: 2217-3412, URL: http://www.ilirias.com
Volume 4 Issue 1 (2013), Pages 1-7.

# A CONTRACTIVE MAPPING THEOREM ON THE $n$-NORMED SPACE OF $p$-SUMMABLE SEQUENCES

S. EKARIANI, H. GUNAWAN, M. IDRIS

Abstract. We revisit the space $\ell^p$ of $p$-summable sequences of real numbers which is equipped with an $n$-norm. We derive a norm from the $n$-norm in a certain way, and show that this norm is equivalent to the usual norm on $\ell^p$. We then use this fact together with others to prove a contractive mapping theorem on the $n$-normed space $\ell^p$.

## 1. Introduction

The theory of 2-normed spaces was first introduced by Gähler [2] in the 1960's. See also [6, 14, 15]. Extensions to the theory of $n$-normed spaces were developed in [3, 4, 5]. Related works may be found in [1, 7, 9, 10, 12, 13]. For a fixed number $n \in \mathbb{N}$, an *$n$-norm* on a real vector space $X$ (of dimension at least $n$) is a mapping $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ which satisfies the following four conditions:

(1.1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,

(1.2) $\|x_1, \dots, x_n\|$ is invariant under permutation,

(1.3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$ for every $\alpha \in \mathbb{R}$,

(1.4) $\|x_1 + x_1', x_2, \dots, x_n\| \le \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an *$n$-normed space*.

Geometrically, $\|x_1, \dots, x_n\|$ may be interpreted as the volume of the $n$-dimensional parallelepiped spanned by $x_1, \dots, x_n$. Note that in an $n$-normed space, we have

$$\|x_1 + y, x_2, \dots, x_n\| = \|x_1, x_2, \dots, x_n\|$$

for any $y = \alpha_2 x_2 + \dots + \alpha_n x_n$.

Let $(X, \|\cdot, \dots, \cdot\|)$ be an $n$-normed space. A sequence $(x_k)$ in $X$ is said to *converge* to an $x \in X$ (in the $n$-norm) if

$$\lim_{k\to\infty} \|x_k - x, y_2, \dots, y_n\| = 0,$$

for every $y_2, \dots, y_n \in X$. Also, a sequence $(x_k)$ in $X$ is called a *Cauchy* sequence if

$$\lim_{k,l\to\infty} \|x_k - x_l, y_2, \dots, y_n\| = 0,$$

---

2000 *Mathematics Subject Classification.* 46B05, 46B20, 46A45, 46B45, 46B99.

*Key words and phrases.* $p$-summable sequences; $n$-normed spaces; norm equivalence; contractive mapping theorem.



Submitted December 18, 2012. Published January 23, 2013.

S. Ekariani and H. Gunawan are supported by ITB I-MHERE PDU Research Grant 2012.

for every $y_2, \dots, y_n \in X$.

If every Cauchy sequence $(x_k)$ in $X$ converges to some $x \in X$, then $X$ is said to be *complete*. A complete $n$-normed space is called an *$n$-Banach Space*.

Our interest here is in the existence of a fixed point of a contractive mapping on an $n$-Banach space. In particular, we shall consider the space $\ell^p$ of $p$-summable sequences of real numbers, where $1 \le p \le \infty$, that is, the space of all sequences $x = (\xi_j)$ for which $\sum\limits_j |\xi_j|^p < \infty$ when $1 \le p < \infty$ (or $\sup\limits_j |\xi_j| < \infty$ when $p = \infty$). For $1 \le p < \infty$, we equip $\ell^p$ with the following $n$-norm (for a fixed number $n \in \mathbb{N}$)

$$\|x_1, \dots, x_n\|_p := \left[ \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \left| \det \left[ \xi_{ij_k} \right]_{i,k} \right|^p \right]^{\frac{1}{p}},$$

where $x_i := (\xi_{ij}) \in \ell^p,\ i = 1, \dots, n$. For $p = \infty$, the $n$-norm is given by

$$\|x_1, \dots, x_n\|_\infty := \sup_{j_1} \cdots \sup_{j_n} \left| \det \left[ \xi_{ij_k} \right]_{i,k} \right|.$$

The $n$-normed space $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ was first studied by Gunawan [8].

In [8], Gunawan formulated a contractive mapping theorem for $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ as follows:

**Theorem 1.1.** [8] *Let $T$ be a self-mapping of $\ell^p$ which is contractive in the sense that there exists a constant $C \in (0, 1)$ such that the inequality*

$$\|Tx - Tx', y_2, \dots, y_n\|_p \le C \, \|x - x', y_2, \dots, y_n\|_p$$

*holds for all $x, x', y_2, \dots, y_n \in \ell^p$. Then $T$ has a unique fixed point, that is, there exists a unique $x \in \ell^p$ such that $Tx = x$.*

We notice that the hypothesis on $T$ in the above theorem is quite strong. The aim of this note is to prove a similar theorem but with a much weaker condition. Additional results will also be presented.

## 2. Main Results

Our main result is the following theorem, which is more general than the previous theorem of Gunawan.

**Theorem 2.1.** *Let $T$ be a self-mapping of $\ell^p$ which is contractive with respect to a linearly independent set $\{a_1, \dots, a_n\}$ in $\ell^p$, that is, there exists a constant $C \in (0, 1)$ such that*

$$\|Tx - Tx', a_{i_2}, \dots, a_{i_n}\|_p \le C \, \|x - x', a_{i_2}, \dots, a_{i_n}\|_p$$

*holds for all $x, x' \in \ell^p$ and $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. Then $T$ has a unique fixed point, that is, there exists a unique $x \in \ell^p$ such that $Tx = x$.*

To prove the above theorem, we need several lemmas and propositions. The first lemma below from [8] gives a relationship between the $n$-norm $\|\cdot, \dots, \cdot\|_p$ and the usual norm $\|\cdot\|_p$ on $\ell^p$. From now on, we assume that $1 \le p < \infty$, and leave the case where $p = \infty$ to the reader.

**Lemma 2.2.** [8] *The inequality*

$$\|x_1, \cdots, x_n\|_p \le (n!)^{1-(1/p)} \|x_1\|_p \cdots \|x_n\|_p$$

*holds for every $x_1, \cdots, x_n \in \ell^p$.*

On an $n$-normed space, we can derive a norm from the $n$-norm in a certain way. In particular, on the space $\ell^p$ of $p$-summable sequences, we have the following result.

**Proposition 2.3.** *Let $\{a_1, \dots, a_n\}$ be a linearly independent set on $\ell^p$. Then, the following function*

$$\|x\|_p^* := \left[ \sum_{\{i_2,\dots,i_n\} \subset \{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}}$$

*defines a norm on $\ell^p$.*

*Proof.* One may verify that $\|x\|_p^*$ satisfies the four properties of a norm. In particular, we may check that if $\|x\|_p^* = 0$, then $\|x, a_{i_2}, \dots, a_{i_n}\|_p = 0$ for every $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. This means that $x$ is in the linear span of $\{a_{i_2}, \dots, a_{i_n}\}$ for every $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. This forces us to conclude that $x = 0$. Furthermore, one may check that the triangle inequality for $\|\cdot\|_p^*$ follows immediately from the triangle inequality for the $n$-norm and the Minkowski's inequality for sums. □

The next lemma is required to show that the norm $\|\cdot\|_p^*$ is actually equivalent to the usual norm $\|\cdot\|_p$ on $\ell^p$.

**Lemma 2.4.** *For every $x, y_1, \dots, y_n \in \ell^p$, we have*

$$\begin{aligned} \|x\|_p \|y_1, \dots, y_n\|_p \le n\|y_1\|_p \|x, y_2, y_3, \dots, y_n\|_p + \\ + \|y_2\|_p \|x, y_1, y_3, \dots, y_n\|_p + \dots + \|y_n\|_p \|x, y_1, y_2, \dots, y_{n-1}\|_p. \end{aligned}$$

*Proof.* The case $n = 2$ has been proved in [11]. Here we shall only give the proof for $n = 3$, leaving the proof for $n > 3$ to the reader. Let $x := (\xi_j) \in \ell^p$ and $y_i := (\eta_{ij}) \in \ell^p,\ i = 1, 2, 3$. We observe that

$$\|x\|_p^p \, \|y_1, y_2, y_3\|_p^p = \frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \xi_k \begin{vmatrix} \eta_{1j_1} & \eta_{1j_2} & \eta_{1j_3} \\ \eta_{2j_1} & \eta_{2j_2} & \eta_{2j_3} \\ \eta_{3j_1} & \eta_{3j_2} & \eta_{3j_3} \end{vmatrix} \right|^p.$$

By Minkowski's inequality, we have

$$\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\xi_k\begin{vmatrix}\eta_{1j_1} & \eta_{1j_2} & \eta_{1j_3}\\ \eta_{2j_1} & \eta_{2j_2} & \eta_{2j_3}\\ \eta_{3j_1} & \eta_{3j_2} & \eta_{3j_3}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}$$

$$\leq\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\eta_{1j_1}\begin{vmatrix}\eta_{2j_2} & \eta_{2_k} & \eta_{2j_3}\\ \xi_{j_2} & \xi_k & \xi_{j_3}\\ \eta_{3j_2} & \eta_{3_k} & \eta_{3j_3}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}+$$

$$\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\eta_{1j_2}\begin{vmatrix}\eta_{2j_1} & \eta_{2_k} & \eta_{2j_3}\\ \eta_{3j_1} & \eta_{3_k} & \eta_{3j_3}\\ \xi_{j_1} & \xi_k & \xi_{j_3}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}+$$

$$\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\eta_{1j_3}\begin{vmatrix}\eta_{2j_1} & \eta_{2_k} & \eta_{2j_2}\\ \xi_{j_1} & \xi_k & \xi_{j_2}\\ \eta_{3j_1} & \eta_{3_k} & \eta_{3j_2}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}+$$

$$\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\eta_{2k}\begin{vmatrix}\eta_{1j_1} & \eta_{1j_3} & \eta_{1j_2}\\ \eta_{3j_1} & \eta_{3j_3} & \eta_{3j_2}\\ \xi_{j_1} & \xi_{j_3} & \xi_{j_2}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}+$$

$$\left(\frac{1}{3!}\sum_k\sum_{j_1}\sum_{j_2}\sum_{j_3}\left|\eta_{3k}\begin{vmatrix}\eta_{1j_1} & \eta_{1j_3} & \eta_{1j_2}\\ \xi_{j_1} & \xi_{j_3} & \xi_{j_2}\\ \eta_{2j_1} & \eta_{2j_3} & \eta_{2j_2}\end{vmatrix}\right|^p\right)^{\frac{1}{p}}$$

$$=3\,\|y_1\|_p\|x,y_2,y_3\|_p+\|y_2\|_p\|x,y_1,y_3\|_p+\|y_3\|_p\|x,y_1,y_2\|_p.$$

Hence we obtain

$$\|x\|_p\|y_1,y_2,y_3\|_p\leq 3\|y_1\|_p\|x,y_2,y_3\|_p+\|y_2\|_p\|x,y_1,y_3\|_p+\|y_3\|_p\|x,y_1,y_2\|_p,$$

as desired. □

**Proposition 2.5.** *Let $\{a_1,\dots,a_n\}$ be a linearly independent set on $\ell^p$. Then the norm $\|\cdot\|_p^*$ defined in Proposotion 2.3 is equivalent to the usual norm $\|\cdot\|_p$ on $\ell^p$. Precisely, we have*

$$\frac{n\|a_1,\dots,a_n\|_p}{(2n-1)\left[\|a_1\|_p+\cdots+\|a_n\|_p\right]}\|x\|_p\leq\|x\|_p^*$$
$$\leq(n!)^{1-\frac{1}{p}}\left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}}\|a_{i_2}\|_p^p\cdots\|a_{i_n}\|_p^p\right]^{\frac{1}{p}}\|x\|_p$$

*for every $x\in\ell^p$.*

*Proof.* For any $x\in\ell^p$ and any subset $\{i_2,\dots,i_n\}$ of $\{1,2,\dots,n\}$, we observe that

$$\|x,a_{i_2},\dots,a_{i_n}\|_p\leq(n!)^{1-(1/p)}\|x\|_p\|a_{i_2}\|_p\cdots\|a_{i_n}\|_p.$$

Hence

$$\begin{aligned}
\|x\|_p^* &= \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p\right]^{\frac{1}{p}} \\
&\le (n!)^{1-\frac{1}{p}} \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p\right]^{\frac{1}{p}} \|x\|_p.
\end{aligned}$$

By the inequality in Lemma 2.4, we have

$$\begin{aligned}
\|x\|_p\|a_1, a_2, \dots, a_n\|_p &\le n\|a_1\|_p\|x, a_2, a_3, \dots, a_n\|_p + \\
&\quad + \|a_2\|_p\|x, a_1, a_3, \dots, a_n\|_p + \\
&\quad \cdots + \|a_n\|_p\|x, a_1, a_2, \dots, a_{n-1}\|_p \\
\\
\|x\|_p\|a_2, a_1, \dots, a_n\|_p &\le n\|a_2\|_p\|x, a_1, a_3, \dots, a_n\|_p + \\
&\quad + \|a_1\|_p\|x, a_2, a_3, \dots, a_n\|_p + \\
&\quad \cdots + \|a_n\|_p\|x, a_1, a_2, \dots, a_{n-1}\|_p \\
&\vdots \\
\|x\|_p\|a_n, a_1, \dots, a_{n-1}\|_p &\le n\|a_n\|_p\|x, a_1, a_2, \dots, a_{n-1}\|_p + \\
&\quad + \|a_1\|_p\|x, a_n, a_2, \dots, a_{n-1}\|_p + \\
&\quad \cdots + \|a_{n-1}\|_p\|x, a_n, a_1, \dots, a_{n-2}\|_p,
\end{aligned}$$

whence

$$\begin{aligned}
n\|x\|_p\|a_1, a_2, \dots, a_n\|_p &\le (2n-1)\|a_1\|_p\|x, a_2, a_3, \dots, a_n\|_p + \\
&\quad \cdots + (2n-1)\|a_n\|_p\|x, a_1, a_2, \dots, a_{n-1}\|_p.
\end{aligned}$$

Next, we observe that

$$\begin{aligned}
\|x, a_2, a_3, \dots, a_n\|_p &\le \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p\right]^{\frac{1}{p}} = \|x\|_p^* \\
&\vdots \\
\|x, a_1, a_2, \dots, a_{n-1}\|_p &\le \left[\sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p\right]^{\frac{1}{p}} = \|x\|_p^*.
\end{aligned}$$

Hence, we obtain

$$n\|x\|_p\|a_1, a_2, \dots, a_n\|_p \le (2n-1)\left[\|a_1\|_p + \cdots + \|a_n\|_p\right]\|x\|_p^*.$$

This completes the proof. □

Since $(\ell^p, \|\cdot\|_p)$ is a Banach space, we have the following corollary.

**Corollary 2.6.** *The normed space $(\ell^p, \|\cdot\|_p^*)$ is a Banach space.*

Now we come to the proof of our main theorem, namely Theorem 2.1. All the previous results make the proof very simple.

*Proof of Theorem 2.1.* For any $x, x' \in \ell^p$, we observe that

$$\begin{aligned}
\|Tx - Tx'\|_p^* &= \left[ \sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|Tx - Tx', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\
&\le \left[ \sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} C^p \|x - x', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\
&= C \left[ \sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x - x', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\
&= C\, \|x - x'\|_p^*.
\end{aligned}$$

This shows that $T$ is a contractive mapping with respect to $\|\cdot\|_p^*$. Since $(\ell^p, \|\cdot\|_p^*)$ is a Banach space, by the contractive mapping theorem for Banach spaces we conclude that $T$ has a unique fixed point in $\ell^p$. ☐

## 3. Further Results

By Lemma 2.2, we see that, in $\ell^p$, if a sequence $(x_k)$ converges to $x$ in the usual norm $\|\cdot\|_p$, then it also converges to $x$ in the $n$-norm $\|\cdot,\dots,\cdot\|_p$. Similarly, if $(x_k)$ is a Cauchy sequence with respect to $\|\cdot\|_p$, then it is also a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|_p$.

Now, as a consequence of Proposition 2.5, we have the following theorem.

**Theorem 3.1.** *A sequence $(x_k) \in \ell^p$ converges to an $x \in \ell^p$ in $\|\cdot,\dots,\cdot\|_p$ if and only if $(x_k)$ converges to $x$ in $\|\cdot\|_p$. Similarly, $(x_k) \in \ell^p$ is a Cauchy sequence with respect to $\|\cdot,\dots,\cdot\|_p$ if and only if $(x_k)$ is a Cauchy sequence with respect to $\|\cdot\|_p$.*

*Proof.* Let $(x_k) \in \ell^p$ converge to an $x \in \ell^p$ in $\|\cdot\|_p$, that is, $\lim_{k\to\infty} \|x_k - x\|_p = 0$. Hence, for every $y_2, \dots, y_n \in \ell^p$, we have (by Lemma 2.2)

$$\lim_{k\to\infty} \|x_k - x, y_2, \dots, y_n\|_p \le \lim_{k\to\infty} (n!)^{1-(1/p)} \|x_k - x\|_p \|y_2\|_p \cdots \|y_n\|_p = 0.$$

This tells us that $(x_k)$ converges to $x$ in $\|\cdot,\dots,\cdot\|_p$.

Conversely, let $(x_k) \in \ell^p$ converge to $x \in \ell^p$ in $\|\cdot,\dots,\cdot\|_p$, that is,

$$\lim_{k\to\infty} \|x_k - x, y_2, \dots, y_n\|_p = 0,$$

for every $y_2, \dots, y_n \in \ell^p$. Let $\{a_1, \dots, a_n\}$ be a linearly independent set on $\ell^p$ and $\|\cdot\|_p^*$ be defined as in Proposition 2.3. Then, for any $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$, we have

$$\lim_{k\to\infty} \|x_k - x, a_{i_2}, \dots, a_{i_n}\|_p = 0,$$

so that $\lim_{k\to\infty} \|x_k - x\|_p^* = 0$. It follows from Proposition 2.5 that $\lim_{k\to\infty} \|x_k - x\|_p = 0$, which means that $(x_k)$ converges to $x$ in $\|\cdot\|_p$.

The proof of the second part is similar, and so we are done. ☐

**Corollary 3.2.** *The $n$-normed space $(\ell^p, \|\cdot,\dots,\cdot\|_p)$ is an $n$-Banach space.*

**Acknowledgments.** The authors would like to thank the anonymous referee for his/her comments that helped us improve this paper.

## References

[1] C. Diminnie, S. Gähler and A. White, *2-inner product spaces*, Demonstratio Math. **6** (1973), 525–536.
[2] S. Gähler, *Lineare 2-normierte räume*, Math. Nachr. **28** (1964), 1–43.
[3] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. I*, Math. Nachr. **40** (1969), 165 - 189.
[4] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. II*, Math. Nachr. **40** (1969), 229–264.
[5] S. Gähler, *Untersuchungen über verallgemeinerte m-metrische Räume. III*, Math. Nachr. **41** (1969), 23–36.
[6] S. Gähler, *Uber 2-Banach räume*, Math. Nachr. **42** (1969), 335–347.
[7] S. Gähler, A.H. Siddiqi and S.C. Gupta, *Contributions to non-Archimedean functional analysis*, Math. Nachr. **69** (1975), 163–171.
[8] H. Gunawan, *The space of p-summable sequences and its natural n-norms*, Bull. Austral. Math. Soc. **64** (2001), 137–147.
[9] H. Gunawan, *On n-inner products, n-norms, and the Cauchy-Schwarz inequality*, Sci. Math. Jpn. **55** (2002), 53–60.
[10] H. Gunawan and Mashadi, *On n-normed spaces*, Int. J. Math. Math. Sci. **27** (2001), 631–639.
[11] M. Idris, S. Ekariani and H. Gunawan, *On the space of p-summable sequences*, Mat. Vesnik. **63** (2013), 58–63.
[12] A. Malčeski, $\ell^\infty$ *as n-normed space*, Mat. Bilten **21** (1997), 103–110.
[13] A. Misiak, *n-inner product spaces*, Math. Nachr. **140** (1989), 299–319.
[14] A.H. Siddiqi, *2-normed spaces*, Aligarh Bull. Math. **7** (1977), 53–70.
[15] A.G. White, *2-Banach spaces*, Math. Nachr. **42** (1969), 43–60.

Shelvi Ekariani
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
*E-mail address*: shelvi_ekariani@students.itb.ac.id.

Hendra Gunawan
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia
*E-mail address*: hgunawan@math.itb.ac.id

Mochammad Idris
Department of Mathematics, Universitas Lambung Mangkurat, Banjarbaru, Indonesia
*E-mail address*: awmius@yahoo.com.