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On Inclusion Properties of Two Versions of Orlicz–Morrey Spaces

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Abstract. There are two versions of Orlicz–Morrey spaces (on $\mathbb{R}^n$), defined by Nakai in 2004 and by Sawano, Sugano, and Tanaka in 2012. In this paper, we discuss the inclusion properties of these two spaces and compare the results. Computing the norms of the characteristic functions of balls in $\mathbb{R}^n$ is one of the keys to our results. Similar results for weak Orlicz–Morrey spaces of both versions are also obtained.

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1. Introduction

Orlicz–Morrey spaces are generalizations of Orlicz spaces and Morrey spaces (on $\mathbb{R}^n$). There are two versions of Orlicz–Morrey spaces: one is defined by Nakai [3,11] and another by Sawano, Sugano, and Tanaka [3,15]. We shall discuss both of them here. In particular, we are interested in the inclusion properties of these spaces.

A function $\Phi : [0,\infty) \to [0,\infty)$ is called a Young function if Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Given two Young functions Φ, Ψ, we write $\Phi \prec \Psi$ if there exists a constant $C > 0$ such that $\Phi(t) \le \Psi(Ct)$ for all $t > 0$.

Let G_1 be the set of all functions $\phi : (0,\infty) \to (0,\infty)$ such that $\phi(r)$ is nondecreasing but $\frac{\phi(r)}{r}$ is nonincreasing. For a Young function Ψ, we also define G_2 to be the set of all functions $\psi : (0,\infty) \to (0,\infty)$ such that $\psi(r)$ is nondecreasing but for any $s > 0$, $\frac{\psi((r+s)^n)}{\Psi^{-1}((\frac{r+s}{s})^n)}$ is nonincreasing.

For $\phi_1,\ \phi_2 : (0,\infty) \to (0,\infty)$, we write $\phi_1 \preceq \phi_2$ if there exists a constant $C > 0$ such that $\phi_1(t) \le C\phi_2(t)$ for all $t > 0$. If $\phi_1 \preceq \phi_2$ and $\phi_2 \preceq \phi_1$, then we write $\phi_1 \approx \phi_2$.

Let Φ be a Young function and $\phi \in G_1$. The Orlicz–Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$ (of Nakai's version) is the set of measurable functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that for every $a \in \mathbb{R}^n$ and $r > 0$, the following quantity

$$\|f\|_{(\phi,\Phi,B(a,r))} := \inf\left\{ b > 0 : \frac{\phi(|B(a,r)|)}{|B(a,r)|} \int_{B(a,r)} \Phi\left(\frac{|f(x)|}{b}\right) \mathrm{d}x \leq 1 \right\}$$

is finite. We use the notation $B(a,r)$ to denote the open ball in $\mathbb{R}^n$ centered at a with radius r, and $|B(a,r)|$ for its Lebesgue measure. The Orlicz–Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|f\|_{L_{\phi,\Phi}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n,\ r>0} \|f\|_{(\phi,\Phi,B(a,r))}$.

For $\phi(r) = r$, the space $L_{\phi,\Phi}(\mathbb{R}^n)$ is the Orlicz space $L_\Phi(\mathbb{R}^n)$. Meanwhile, for $\Phi(r) = r^p$ and $\phi(r) = r^{1-\frac{\lambda}{n}}$ where $0 \leq \lambda \leq n$, the space $L_{\phi,\Phi}(\mathbb{R}^n)$ reduces to the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$.

Now, let Ψ be a Young function and $\psi \in G_2$. Sawano, Sugano, and Tanaka defined the Orlicz–Morrey space $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ to be the set of measurable functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n,\ r>0} \psi(|B(a,r)|)\|f\|_{(\Psi,B(a,r))} < \infty,$$

where $\|f\|_{(\Psi,B(a,r))} := \inf\left\{b > 0 : \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi\left(\frac{|f(x)|}{b}\right)\mathrm{d}x \leq 1\right\}$. Notice here that $\psi(|B(a,r)|)$ dominates of the growth $\|f\|_{(\Psi,B(a,r))}$.

For the Young function $\Psi(t) = t^p$ $(1 \leq p < \infty)$, the spaces $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ are recognized as the generalized Morrey spaces $\mathcal{M}^p_\psi(\mathbb{R}^n)$.

Recently, Gunawan et al. [5] presented a sufficient and necessary condition for the inclusion relation between generalized Morrey spaces, as in the following theorem.

Theorem 1.1. *Let $1 \leq p_1 \leq p_2 < \infty$ and $\psi_1, \psi_2 \in G_2$. Then the following statements are equivalent:*

(a) $\psi_1 \preceq \psi_2$.
(b) $\mathcal{M}^{p_2}_{\psi_2}(\mathbb{R}^n) \subseteq \mathcal{M}^{p_1}_{\psi_1}(\mathbb{R}^n)$.
(c) *There exists a constant $C > 0$ such that $\|f\|_{\mathcal{M}^{p_1}_{\psi_1}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}^{p_2}_{\psi_2}(\mathbb{R}^n)}$ for every $f \in \mathcal{M}^{p_2}_{\psi_2}(\mathbb{R}^n)$.*

In the same paper, Gunawan et al. also gave a necessary and sufficient condition for the inclusion relation between generalized weak Morrey spaces.

Meanwhile, the inclusion relation between Orlicz spaces $L_\Phi(\mathbb{R}^n)$ and between weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ is known (see [8,9]). In 2016, Masta et al. [10] also obtained the inclusion properties of Orlicz–Morrey space $L_{\phi,\Phi}(\mathbb{R}^n)$ of Nakai's version, as in the following theorem.

Theorem 1.2. *Let Φ_1, Φ_2 be Young functions and $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \approx \phi_2$. Then the following statements are equivalent:*

(1) $\Phi_1 \prec \Phi_2$.
(2) $L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that*

$$\|f\|_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)},$$

for every $f \in L_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Remark 1.3. Note that the relation $\Phi_1 \prec \Phi_2$ is a necessary and sufficient condition for the inclusion relation between Orlicz–Morrey spaces of Nakai's version. For $\phi_1(t) = \phi_2(t) = t$, Theorem 1.2 reduces to Theorem 3.4(a) in [8]. Furthermore, for $\phi_1(t) = \phi_2(t) = t$ and $w_1(x) = w_2(x) = 1$, Theorem 1.2 complements Corollary 2.11 in [13], which states that $\Phi_1 \prec \Phi_2$ is a sufficient condition for inclusion relation between Orlicz spaces. Related results about inclusion properties of Orlicz–Morrey spaces can be found in [6].

In this paper, we would like to obtain the inclusion properties of Orlicz–Morrey spaces $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ of Sawano–Sugano–Tanaka's version, and compare it with the result for Nakai's version. In addition, we will also prove similar results for weak Orlicz–Morrey spaces of both versions. With our results, we can see what parameters are significant in the inclusion properties for both versions.

To prove the results, we will use the same method as in [5,9,10], that is, by computing the norms of the characteristic functions of balls in $\mathbb{R}^n$. We also employ the properties of the inverse function of Φ, which are presented in the following lemma.

Lemma 1.4. [10,11,14] *Suppose that* Φ *is a Young function and* Φ^{-1} *denotes its inverse, which is given by* $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$ *for every* $s \geq 0$. *Then the following hold:*

(1) $\Phi^{-1}(0) = 0$.
(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ *for* $s_1 \leq s_2$.
(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.
(4) *If, for some constants* $C_1, C_2 > 0$, *we have* $\Phi_2^{-1}(s) \leq C_1\Phi_1^{-1}(C_2 s)$, *then* $\Phi_1(\frac{t}{C_1}) \leq C_2\Phi_2(t)$ *for* $t = \Phi_2^{-1}(s)$.

Throughout this paper, the letter C denotes a constant that may vary in values from line to line. To keep track of some constants, we use subscripts, such as C_1 and C_2.

2. Inclusion Properties of Orlicz–Morrey Spaces

As mentioned earlier, the key to our results is knowing the norms of the characteristic balls in $\mathbb{R}^n$. Here is the first one on $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$:

Lemma 2.1. [3] *For every* $a \in \mathbb{R}^n$ *and* $r > 0$, *we have* $\|\chi_{B(a,r)}\|_{\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r)|)}{\Psi^{-1}(1)}$.

Our first theorem gives equivalent statements for the inclusion relation between Orlicz–Morrey spaces of Sawano–Sugano–Tanaka's version.

Theorem 2.2. *Let* Ψ_1, Ψ_2 *be Young functions such that* $\Psi_1 \prec \Psi_2$ *and* $\psi_1, \psi_2 \in G_2$. *Then the following statements are equivalent:*

(1) $\psi_1 \preceq \psi_2$.
(2) $\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq \mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

(3) *There exists a constant* $C > 0$ *such that*

$$\|f\|_{\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)}$$

for every $f \in \mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)$.

Proof. Let us first prove that (1) implies (2). Let $f \in \mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)$. Recall that $\Psi_1 \prec \Psi_2$ means that there exists a constant $C_1 > 0$ such that $\Psi_1(t) \leq \Psi_2(C_1 t)$ for every $t > 0$. For every $a \in \mathbb{R}^n$ and $r > 0$, let $A_{(\Psi_1,B(a,r))} = \left\{b > 0 : \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_1\left(\frac{|f(x)|}{C_1 b}\right)\mathrm{d}x \leq 1\right\}$ and $A_{(\Psi_2,B(a,r))} = \left\{b > 0 : \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{|f(x)|}{b}\right)\mathrm{d}x \leq 1\right\}$. Thus, for any $b \in A_{(\Psi_2,B(a,r))}$, we have

$$\begin{aligned}\frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_1\left(\frac{|f(x)|}{C_1 b}\right)\mathrm{d}x &\leq \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{C_1|f(x)|}{C_1 b}\right)\mathrm{d}x\\ &= \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{|f(x)|}{b}\right)\mathrm{d}x \leq 1.\end{aligned}$$

Hence, it follows that $b \in A_{(\Psi_1,B(a,r))}$, and so we conclude that $A_{(\Psi_2,B(a,r))} \subseteq A_{(\Psi_1,B(a,r))}$. Accordingly, we have

$$\left\|\frac{f}{C_1}\right\|_{(\Psi_1,B(a,r))} = \inf A_{(\Psi_1,B(a,r))} \leq \inf A_{(\Psi_2,B(a,r))} = \|f\|_{(\Psi_2,B(a,r))},$$

and this holds for every $a \in \mathbb{R}^n$ and $r > 0$.

Now there exists $C_2 > 0$ such that $\psi_1(s) \leq C_2\psi_2(s)$ for every $s > 0$. Combining this with the previous estimate, we obtain

$$\begin{aligned}\|f\|_{\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n,\ r>0} \psi_1(|B(a,r)|)\|f\|_{(\Psi_1,B(a,r))}\\ &\leq \sup_{a\in\mathbb{R}^n,\ r>0} C_1C_2\psi_2(|B(a,r)|)\|f\|_{(\Psi_2,B(a,r))}\\ &= C\|f\|_{\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)}.\end{aligned}$$

This proves that $\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq \mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

Next, since $(\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n), \mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n))$ is a Banach pair, it follows from [7, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume that (3) holds. Let $a \in \mathbb{R}^n$ and $r > 0$. By Lemma 2.1, we have

$$\begin{aligned}\frac{\psi_1(|B(a,r)|)}{\Psi_1^{-1}(1)} &= \|\chi_{B(a,r)}\|_{\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)} \leq C\|\chi_{B(a,r)}\|_{\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)}\\ &= \frac{C\psi_2(|B(a,r)|)}{\Psi_2^{-1}(1)},\end{aligned}$$

whence $\psi_1(|B(a,r)|) \leq \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}\psi_2(|B(a,r)|)$. Since $a \in \mathbb{R}^n$ and $r > 0$ are arbitrary, we get $\psi_1(t) \leq C_1\psi_2(t)$ for every $t > 0$, where $C_1 = \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}$. $\square$

Corollary 2.3. *Let* Ψ *be a Young function and* $\psi_1, \psi_2 \in G_2$. *Then the following statements are equivalent:*

(1) $\psi_1 \preceq \psi_2$.
(2) $\mathcal{M}_{\psi_2,\Psi}(\mathbb{R}^n) \subseteq \mathcal{M}_{\psi_1,\Psi}(\mathbb{R}^n)$.

(3) *There exists a constant* $C > 0$ *such that*

$$\|f\|_{\mathcal{M}_{\psi_1,\Psi}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_{\psi_2,\Psi}(\mathbb{R}^n)}$$

for every $f \in \mathcal{M}_{\psi_2,\Psi}(\mathbb{R}^n)$.

Remark 2.4. We note that the relation $\psi_1 \preceq \psi_2$ is a necessary and sufficient condition for the inclusion relation between Orlicz–Morrey spaces of Sawano–Sugano–Tanaka's version.

3. Inclusion Properties of Weak Orlicz–Morrey Spaces

We shall now discuss the inclusion properties of weak Orlicz–Morrey spaces. First, we recall the definition of weak Orlicz–Morrey spaces $wL_{\phi,\Phi}(\mathbb{R}^n)$ [12]. Let Φ be a Young function and $\phi \in G_1$. The weak Orlicz–Morrey space $wL_{\phi,\Phi}(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n, r>0} \|f\|_{wL_{\phi,\Phi,B(a,r)}} < \infty$, where

$$\|f\|_{wL_{\phi,\Phi,B(a,r)}} := \inf\left\{ b > 0 : \sup_{t>0} \frac{\Phi(t)\phi(|B(a,r)|)\left|\{x \in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \le 1 \right\}$$

for $a \in \mathbb{R}^n$ and $r > 0$. If $\Phi(t) = t^p$, $1 \le p < \infty$ and $\phi(r) = r$, the space $wL_{\phi,\Phi}(\mathbb{R}^n)$ is the weak Lebesgue space $wL_p(\mathbb{R}^n)$ (see [1]).

The relation between $wL_{\phi,\Phi}(\mathbb{R}^n)$ and $L_{\phi,\Phi}(\mathbb{R}^n)$ is presented in the following lemma. (We leave the proof to the reader.)

Lemma 3.1. *Let* Φ *be a Young function and* $\phi \in G_1$. *Then* $L_{\phi,\Phi}(\mathbb{R}^n) \subseteq wL_{\phi,\Phi}(\mathbb{R}^n)$ *with* $\|f\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} \le \|f\|_{L_{\phi,\Phi}(\mathbb{R}^n)}$ *for every* $f \in L_{\phi,\Phi}(\mathbb{R}^n)$.

The following lemma gives the norms of the characteristic functions of balls in $\mathbb{R}^n$.

Lemma 3.2. *Let* Φ *be a Young function,* $\phi \in G_1$, $a \in \mathbb{R}^n$, *and* $r, r_0 > 0$. *Then we have*

$$\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi,B(a,r)}} = \frac{1}{\Phi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\right)}.$$

Proof. Since $\|\cdot\|_{wL_{\phi,\Phi,B(a,r)}} \le \|\cdot\|_{(\phi,\Phi,B(a,r))}$ and $\|\chi_{B(a,r_0)}\|_{(\phi,\Phi,B(a,r))} = \frac{1}{\Phi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\right)}$, we obtain

$$\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi,B(a,r)}} \le \frac{1}{\Phi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\right)}.$$

By the definitions of Φ^{-1} and $\|\cdot\|_{wL_{\phi,\Phi,B(a,r)}}$, we conclude that

$$\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi,B(a,r)}} = \frac{1}{\Phi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\right)}.$$

□

Lemma 3.3. *Let Φ be a Young function, $\phi \in G_1$, $a \in \mathbb{R}^n$, and $r_0 > 0$ be arbitrary. Then we have* $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$.

Proof. Since Φ is a Young function and $\phi \in G_1$, we have $\|\chi_{B(a,r_0)}\|_{L_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$ for $a \in \mathbb{R}^n$ and $r_0 > 0$ (see [3]). Hence, by Lemma 3.1, we have $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} \leq \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$. On the other hand,

$$\begin{aligned}\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n, r>0} \|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi,B(a,r)}} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\Phi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\right)} \\ &\geq \frac{1}{\Phi^{-1}\left(\frac{1}{\phi(|B(a,r_0)|)}\right)}.\end{aligned}$$

Consequently, we have $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$. □

Now we come to the inclusion property of weak Orlicz–Morrey spaces of Nakai's version.

Theorem 3.4. *Let Φ_1, Φ_2 be Young functions, $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \preceq \phi_2$. Then the following statements are equivalent:*

(1) $\Phi_1 \prec \Phi_2$.
(2) $wL_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq wL_{\phi_1,\Phi_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that*

$$\|f\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C\|f\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)}$$

for every $f \in wL_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Proof. Assume that (1) holds and let $f \in wL_{\phi_2,\Phi_2}(\mathbb{R}^n)$. Since $\Phi_1 \prec \Phi_2$ and $\phi_1 \preceq \phi_2$, there exist constants $C_1, C_2 > 0$ such that $\Phi_1(t) \leq \Phi_2(C_1 t)$ and $\phi_1(t) \leq C_2\phi_2(t)$ for every $t > 0$. Let $a \in \mathbb{R}^n$ and $r > 0$. We consider two cases.

Case I: $C_2 \geq 1$. Let

$$\begin{aligned}&A_{\phi_1,\Phi_1,B(a,r)} \\ &= \left\{ b > 0 : \sup_{t>0} \frac{\Phi_1(\frac{t}{C_2})\phi_1(|B(a,r)|)\left|\{x \in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \leq 1 \right\} \\ &= \left\{ b > 0 : \sup_{t_1>0} \frac{\Phi_1(t_1)\phi_1(|B(a,r)|)\left|\{x \in B(a,r) : \frac{|f(x)|}{b} > C_2 t_1\}\right|}{|B(a,r)|} \leq 1 \right\} \\ &= \left\{ b > 0 : \sup_{t_1>0} \frac{\Phi_1(t_1)\phi_1(|B(a,r)|)\left|\{x \in B(a,r) : \frac{|f(x)|}{C_2 b} > t_1\}\right|}{|B(a,r)|} \leq 1 \right\}\end{aligned}$$

and

$$\begin{aligned} &A_{\phi_2,\Phi_2,B(a,r)} \\ &= \left\{ b>0 : \sup_{t>0} \frac{\Phi_2(C_1 t)\phi_2(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \le 1 \right\} \\ &= \left\{ b>0 : \sup_{t_1>0} \frac{\Phi_2(t_1)\phi_2(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > \frac{t_1}{C_1}\}\right|}{|B(a,r)|} \le 1 \right\} \\ &= \left\{ b>0 : \sup_{t_1>0} \frac{\Phi_2(t_1)\phi_2(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|C_1 f(x)|}{b} > t_1\}\right|}{|B(a,r)|} \le 1 \right\}. \end{aligned}$$

Then $\left\|\frac{f}{C_2}\right\|_{wL_{\phi_1,\Phi_1,B(a,r)}} = \inf A_{\phi_1,\Phi_1,B(a,r)}$ and $\|C_1 f\|_{wL_{\phi_2,\Phi_2,B(a,r)}} = \inf A_{\phi_2,\Phi_2,B(a,r)}$. Observe that, for arbitrary $b\in A_{\phi_2,\Phi_2,B(a,r)}$ and $t>0$, we have

$$\begin{aligned} &\frac{\Phi_1(\frac{t}{C_2})\phi_1(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \\ &\le \frac{\Phi_1(t)\frac{\phi_1(|B(a,r)|)}{C_2}\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \\ &\le \frac{\Phi_2(C_1 t)\phi_2(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \\ &\le \sup_{t_1>0} \frac{\Phi_2(t_1)\phi_2(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|C_1 f(x)|}{b} > t_1\}\right|}{|B(a,r)|} \\ &\le 1. \end{aligned}$$

Since $t>0$ is arbitrary, we have $\sup_{t>0} \frac{\Phi_1(\frac{t}{C_2})\phi_1(|B(a,r)|)\left|\{x\in B(a,r): \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \le 1$. Hence, it follows that $b\in A_{\phi_1,\Phi_1,B(a,r)}$, and so we conclude that $A_{\phi_2,\Phi_2,B(a,r)} \subseteq A_{\phi_1,\Phi_1,B(a,r)}$. Accordingly, we obtain

$$\begin{aligned} \left\|\frac{f}{C_2}\right\|_{wL_{\phi_1,\Phi_1,B(a,r)}} &= \inf A_{\phi_1,\Phi_1,B(a,r)} \\ &\le \inf A_{\phi_2,\Phi_2,B(a,r)} = \|C_1 f\|_{wL_{\phi_2,\Phi_2,B(a,r)}}. \end{aligned}$$

Case II: $0<C_2<1$. Observe that, for arbitrary $b\in A_{(\phi_2,\Phi_2,B(a,r))}$ and $t>0$, we have (by setting $s=\frac{t}{C_2}$)

$$\begin{aligned} &\frac{\Phi_1(t)\phi_1(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|C_2 f(x)|}{b} > t\}\right|}{|B(a,r)|} \\ &= \frac{\Phi_1(t)\phi_1(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > \frac{t}{C_2}\}\right|}{|B(a,r)|} \\ &= \frac{\Phi_1(C_2 s)\phi_1(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > s\}\right|}{|B(a,r)|} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_2\Phi_1(s)\phi_1(|B(a,r)|)\big|\{x\in B(a,r): \frac{|f(x)|}{b}>s\}\big|}{|B(a,r)|}\\
&\leq \frac{C_2^2\Phi_2(C_1s)\phi_2(|B(a,r)|)\big|\{x\in B(a,r): \frac{|f(x)|}{b}>s\}\big|}{|B(a,r)|}\\
&\leq \frac{\Phi_2(C_1s)\phi_2(|B(a,r)|)\big|\{x\in B(a,r): \frac{|f(x)|}{b}>s\}\big|}{|B(a,r)|}\\
&\leq \sup_{t_1>0} \frac{\Phi_2(t_1)\phi_2(|B(a,r)|)\big|\{x\in B(a,r): \frac{|C_1f(x)|}{b}>t_1\}\big|}{|B(a,r)|}\\
&\leq 1.
\end{aligned}$$

Since $t>0$ is arbitrary, we have $\sup_{t>0}\frac{\Phi_1(t)\phi_1(|B(a,r)|)\Big|\{x\in B(a,r):\frac{|C_2f(x)|}{b}>t\}\Big|}{|B(a,r)|}\leq 1$. Accordingly, we obtain $\|C_2f\|_{wL_{\phi_1,\Phi_1,B(a,r)}}\leq \|C_1f\|_{wL_{\phi_2,\Phi_2,B(a,r)}}$.

From Cases I and II, there exists a constant $C>0$ such that $\|f\|_{wL_{\phi_1,\Phi_1,B(a,r)}}\leq C\|f\|_{wL_{\phi_2,\Phi_2,B(a,r)}}$. Since $a\in\mathbb{R}^n$ and $r>0$ are arbitrary, we conclude that $\|f\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)}\leq C\|f\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)}$, which implies that $wL_{\phi_2,\Phi_2}(\mathbb{R}^n)\subseteq wL_{\phi_1,\Phi_1}(\mathbb{R}^n)$.

As mentioned in [12, Appendix G], we know that Lemma 3.3 in [7] still holds for quasi-Banach spaces, so (2) and (3) are equivalent.

Now, we will show that (3) implies (1). To do so, assume that (3) holds. By Lemma 3.3, we have

$$\begin{aligned}
\frac{1}{\Phi_1^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)})} &= \|\chi_{B(a,r_0)}\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)}\\
&\leq C\|\chi_{B(a,r_0)}\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})},
\end{aligned}$$

whence $\Phi_2^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)})\leq C\Phi_1^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})\leq C\Phi_1^{-1}(\frac{C_2}{\phi_1(|B(a,r_0)|)})$, for arbitrary $a\in\mathbb{R}^n$ and $r_0>0$. By Lemma 1.4(4), we have

$$\Phi_1\Big(\frac{t_0}{C}\Big)\leq C_2\Phi_2(t_0),$$

where $t_0=\Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})$. If $C_2\leq 1$, then $\Phi_1(\frac{t_0}{C})\leq\Phi_2(t_0)$. If $C_2>1$, then noting that Φ_1 is convex, we have

$$\Phi_1\Big(\frac{t_0}{C_2C}\Big)\leq\frac{1}{C_2}\Phi_1\Big(\frac{t_0}{C}\Big)\leq\Phi_2(t_0).$$

Since $a\in\mathbb{R}^n$ and $r_0>0$ are arbitrary, we conclude that there exists $C_3>0$ such that $\Phi_1(\frac{t}{C_3})\leq\Phi_2(t)$ or equivalently $\Phi_1(t)\leq\Phi_2(C_3t)$ for every $t>0$. $\square$

Remark 3.5. For $\phi_1(t)=\phi_2(t)=t$, Theorem 3.4 reduces to Theorem 3.3 in [9].

We shall now study the inclusion properties of weak Orlicz–Morrey spaces $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$. Let Ψ be a Young function and $\psi\in G_2$. The weak Orlicz–Morrey space $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ is the set of all measurable functions $f:\mathbb{R}^n\to\mathbb{R}$ such that

$$\|f\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} = \sup_{a\in\mathbb{R}^n, r>0} \psi(|B(a,r)|)\|f\|_{w\mathcal{M}_{\Psi,B(a,r)}} < \infty,$$

where

$$\|f\|_{w\mathcal{M}_{\Psi,B(a,r)}} := \inf\left\{ b>0 : \sup_{t>0} \frac{\Psi(t)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \le 1 \right\},$$

for $a\in\mathbb{R}^n$ and $r>0$. Note that if there exists $C>0$ such that $\Psi_1(t) \le \Psi_2(Ct)$ for every $t>0$, then $\|f\|_{w\mathcal{M}_{\Psi_1,B(a,r)}} \le C\|f\|_{w\mathcal{M}_{\Psi_2,B(a,r)}}$ for every $a\in\mathbb{R}^n$ and $r>0$.

The following lemma tells us that $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ contains $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$. (We leave the proof to the reader.)

Lemma 3.6. *Let Ψ be a Young function and $\psi \in G_2$. Then $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ with $\|f\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} \le \|f\|_{\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)}$ for every $f\in\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$.*

Following similar arguments as in the proof of Lemma 3.2, we have the following lemma.

Lemma 3.7. *Let Ψ be a Young function, $a\in\mathbb{R}^n$, and $r, r_0 > 0$. Then we have*

$$\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\Psi,B(a,r)}} = \frac{1}{\Psi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|}\right)}.$$

The norms of the characteristic functions of balls in $\mathbb{R}^n$ are presented in the following lemma.

Lemma 3.8. *Let Ψ be a Young function, $\psi\in G_2$, $a\in\mathbb{R}^n$, and $r_0>0$ be arbitrary. Then we have $\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$.*

Proof. Since Ψ is a Young function and $\psi\in G_2$, by Lemmas 2.1 and 3.6 we have $\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} \le \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$, for $a\in\mathbb{R}^n$ and $r_0>0$. On the other hand, we have

$$\begin{aligned}\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n, r>0} \psi(|B(a,r)|)\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\Psi,B(a,r)}} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{\psi(|B(a,r)|)}{\Psi^{-1}\left(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|}\right)} \ge \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}.\end{aligned}$$

Consequently, we have $\|\chi_{B(a,r_0)}\|_{w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$, as desired. □

Now we come to the inclusion property of weak Orlicz–Morrey spaces of Sawano–Sugano–Tanaka's version.

Theorem 3.9. *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \prec \Psi_2$ and $\psi_1, \psi_2 \in G_2$. Then the following statements are equivalent:*

(1) $\psi_1 \preceq \psi_2$.
(2) $w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C>0$ such that*

$$\|f\|_{w\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)} \le \|f\|_{w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)}$$

for every $f\in w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)$.

Proof. Assume that (1) holds. Let $f \in w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)$. Since $\Psi_1 \prec \Psi_2$ and $\psi_1 \preceq \psi_2$, there exist constant $C_1, C_2 > 0$ such that $\psi_1(t) \le C_1\psi_2(t)$ and $\Psi_1(t) \le \Psi_2(C_2 t)$ for every $t > 0$. Observe that

$$\begin{aligned}\|f\|_{w\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n, r>0} \psi_1(|B(a,r)|)\|f\|_{w\mathcal{M}_{\Psi_1,B(a,r)}}\\ &\le \sup_{a\in\mathbb{R}^n, r>0} C_1\psi_2(|B(a,r)|)\|f\|_{w\mathcal{M}_{\Psi_1,B(a,r)}}\\ &\le \sup_{a\in\mathbb{R}^n, r>0} C_1C_2\psi_2(|B(a,r)|)\|f\|_{w\mathcal{M}_{\Psi_2,B(a,r)}}\\ &= C_1C_2\|f\|_{w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)}.\end{aligned}$$

Hence, we conclude that $w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq w\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

As before, (2) and (3) are equivalent, and so it remains to show that (3) implies (1). To do so, assume that (3) holds. By Lemma 3.8, we have

$$\begin{aligned}\frac{\psi_1(|B(a,r)|)}{\Psi_1^{-1}(1)} &= \|\chi_{B(a,r)}\|_{w\mathcal{M}_{\psi_1,\Psi_1}(\mathbb{R}^n)}\\ &\le C\|\chi_{B(a,r)}\|_{w\mathcal{M}_{\psi_2,\Psi_2}(\mathbb{R}^n)} = \frac{C\psi_2(|B(a,r)|)}{\Psi_2^{-1}(1)},\end{aligned}$$

whence $\psi_1(|B(a,r)|) \le \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}\psi_2(|B(a,r)|)$, for every $a \in \mathbb{R}^n$ and $r > 0$. We conclude that

$$\psi_1(t) \le C_1\psi_2(t)$$

for every $t > 0$, with $C_1 = \frac{C\Psi_1^{-1}(1)}{\Psi_2^{-1}(1)}$. □

4. Further Results and Concluding Remarks

The inclusion properties of Orlicz–Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$ (Theorem 1.2) and weak Orlicz–Morrey spaces $wL_{\phi,\Phi}(\mathbb{R}^n)$ (Theorem 3.4) generalize the inclusion properties of Orlicz spaces and weak Orlicz spaces in [8,9]. Meanwhile, the inclusion properties of Orlicz–Morrey spaces $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ (Theorem 2.2) and weak Orlicz–Morrey spaces $w\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ (Theorem 3.9) generalize the inclusion properties of generalized Morrey spaces and generalized weak Morrey spaces in [5]. Combining the results, one realizes that the inclusion relation between Orlicz–Morrey spaces is equivalent to that between weak Orlicz–Morrey spaces.

Recently, Guliyev, et al. [2,4] also introduced (strong) Orlicz–Morrey spaces different from Nakai's or Sawano–Sugano–Tanaka's versions. For a Young function Θ, let G_Θ be the set of all functions $\theta : (0,\infty) \to (0,\infty)$ such that $\theta(r)$ is decreasing but $\Theta^{-1}(t^{-n})\theta(t)^{-1}$ is almost decreasing for all $t > 0$. Now, let Θ be a Young function and $\theta \in G_\Theta$. As in [4], may they define the Orlicz–Morrey space $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ to be the set of measurable functions f such that

$$\|f\|_{\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)} := \sup_{a\in\mathbb{R}^n,\ r>0} \frac{1}{\theta(|B(a,r)|^{\frac{1}{n}})}\Theta^{-1}\Big(\frac{1}{|B(a,r)|}\Big)\|f\|_{L_\Theta(B(a,r))} < \infty,$$

where $\|f\|_{L_\Theta(B(a,r))} := \inf\big\{b>0 : \int_{B(a,r)} \Theta\big(\frac{|f(x)|}{b}\big)\mathrm{d}x \le 1\big\}$.

Using similar arguments in the proof of Theorems 1.2 and 2.2, one may obtain the following theorem.

Theorem 4.1. *Let Θ_1, Θ_2 be Young functions such that $\Theta_1 \prec \Theta_2$, $\Theta_1^{-1} \prec \Theta_2^{-1}$, and $\theta_1, \theta_2 \in G_\Theta$. Then the following statements are equivalent:*

(1) $\theta_2 \preceq \theta_1$.
(2) $\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n) \subseteq \mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)$.
(3) *There exists a constant $C > 0$ such that*

$$\|f\|_{\mathcal{M}_{\theta_1,\Theta_1}(\mathbb{R}^n)} \le C\|f\|_{\mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)}$$

for every $f \in \mathcal{M}_{\theta_2,\Theta_2}(\mathbb{R}^n)$.

Comparing Theorems 2.2 and 4.1, we can say that the condition on the growth parameters for the inclusion of Orlicz–Morrey spaces $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ and $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$ are in principal the same. However, the condition on the Young function for the inclusion of the Orlicz–Morrey space $\mathcal{M}_{\psi,\Psi}(\mathbb{R}^n)$ is simpler than that for the Orlicz–Morrey space $\mathcal{M}_{\theta,\Theta}(\mathbb{R}^n)$.

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