SOOCHOW JOURNAL OF MATHEMATICS
Volume 28, No. 4, pp. 389-398, October 2002

INNER PRODUCTS ON n-INNER PRODUCT SPACES

BY

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Abstract. In this note, we show that in any n-inner product space with $n \geq 2$ we can explicitly derive an inner product or, more generally, an $(n-k)$-inner product from the n-inner product, for each $k \in \{1, \ldots, n-1\}$. We also present some related results on n-normed spaces.

1. Introduction

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X^{n+1} satisfying the following five properties:

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0; \langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(I3) $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle, \ \alpha \in \mathbf{R}$;

(I5) $\langle x + x', y | x_2, \ldots, x_n \rangle = \langle x, y | x_2, \ldots, x_n \rangle + \langle x', y | x_2, \ldots, x_n \rangle$,

is called an *n-inner product* on X, and the pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an *n-inner product space.*

For $n = 1$, the expression $\langle x, y | x_2, \ldots, x_n \rangle$ is to be understood as $\langle x, y \rangle$, which denotes nothing but an inner product on X. The concept of 2-inner product spaces was first introduced by Diminnie, Gähler and White [2, 3, 7] in 1970's,

Received March 26, 2001; revised August 13, 2002.
AMS Subject Classification. 46C50, 46B20, 46B99, 46A19.
Key words. n-inner product spaces, n-normed spaces, inner product spaces.

while its generalization for $n \geq 2$ was developed by Misiak [12] in 1989. Note here that our definition of n-inner products is slightly simpler than, but equivalent to, that in [12].

On an n-inner product space $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$, one may observe that the following function

$$\|x_1, x_2, \dots, x_n\| := \langle x_1, x_1|x_2, \dots, x_n\rangle^{1/2},$$

defines an n-norm, which enjoys the following four properties:

(N1) $\|x_1, \dots, x_n\| \geq 0$; $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha|\, \|x_1, x_2, \dots, x_n\|,\ \alpha \in \mathbf{R}$;

(N4) $\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$.

Just as in an inner product space, we have the Cauchy-Schwarz inequality:

$$|\langle x, y|x_2, \dots, x_n\rangle| \leq \|x, x_2, \dots, x_n\|\, \|y, x_2, \dots, x_n\|,$$

and the equality holds if and only if $x, y, x_2, \dots, x_n$ are linearly dependent (see [9]). Furthermore, we have the polarization identity:

$$\|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2 = 4\langle x, y|x_2, \dots, x_n\rangle,$$

and the parallelogram law:

$$\|x+y, x_2, \dots, x_n\|^2 + \|x-y, x_2, \dots, x_n\|^2 = 2(\|x, x_2, \dots, x_n\|^2 + \|y, x_2, \dots, x_n\|^2).$$

The latter gives a characterization of n-inner product spaces.

By the polarization identity and the property (I2), one may observe that

$$\langle x, y|x_2, \dots, x_n\rangle = \langle x, y|x_{i_2}, \dots, x_{i_n}\rangle,$$

for every permutation $(i_2, \dots, i_n)$ of $(2, \dots, n)$. Moreover, one can also show that

$$\langle x, y|x_2, \dots, x_n\rangle = 0,$$

when x or y is a linear combination of $x_2, \dots, x_n$, or when $x_2, \dots, x_n$ are linearly dependent.

Now, for example, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then the following function

$$\langle x, y|x_2, \dots, x_n\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, x_2\rangle & \cdots & \langle x, x_n\rangle \\ \langle x_2, y\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, y\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix},$$

defines an n-inner product, called the standard (or simple) n-inner product on X. Its induced n-norm $\|x_1, x_2, \dots, x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, \dots, x_n$.

Historically, the concept of n-norms were introduced earlier by Gähler in order to generalize the notion of length, area and volume in a real vector space (see [4, 5, 6]). The objects studied here are n-dimensional parallelepipeds. The concept of n-inner products is thus useful when one talks about the angle between two n-dimensional parallelepipeds having the same $(n-1)$-dimensional base.

In this note, we shall show that in any n-inner product space with $n \geq 2$ we can derive an $(n-k)$-inner product from the n-inner product for each $k \in \{1, \dots, n-1\}$. In particular, in any n-inner product space, we can derive an inner product from the n-inner product, so that one can talk about, for instance, the angle between two vectors, as one might like to.

In addition, we shall present some related results on n-normed spaces. See [5] and [11] for previous results on these spaces.

2. Main Results

To avoid confusion, we shall sometimes denote an n-inner product by $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle_n$ and an n-norm by $\|\cdot,\dots,\cdot\|_n$.

Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle_n)$ be an n-inner product space with $n \geq 2$. Fix a linearly independent set $\{a_1, \dots, a_n\}$ in X. With respect to $\{a_1, \dots, a_n\}$, define for each $k \in \{1, \dots, n-1\}$ the function $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle_{n-k}$ on X^{n-k+1} by

$$\langle x, y|x_2, \dots, x_{n-k}\rangle := \sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}} \langle x, y|x_2, \dots, x_{n-k}, a_{i_1}, \dots, a_{i_k}\rangle.$$

Then we have the following fact:

Fact 2.1. *For every* $k \in \{1, \ldots, n-1\}$, *the function* $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_{n-k}$ *defines an* $(n-k)$*-inner product on* X. *In particular, when* $k = n-1$,

$$\langle x, y \rangle := \sum_{\{i_2,\ldots,i_n\} \subseteq \{1,\ldots,n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle, \tag{1}$$

defines an inner product on X.

Proof. It is not hard to see that the function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_{n-k}$ satisfies the five properties (I1)–(I5) of an $(n-k)$-inner product, except perhaps to establish the second part of (I1). To verify this property, suppose that $x_1, \ldots, x_{n-k}$ are linearly dependent. Then $\langle x_1, x_1 | x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle = 0$ for every $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, and hence $\langle x_1, x_1 | x_2, \ldots, x_{n-k} \rangle = 0$. Conversely, suppose that $\langle x_1, x_1 | x_2, \ldots, x_{n-k} \rangle = 0$. Then $\langle x_1, x_1 | x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k} \rangle = 0$ for every $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. Hence $\{x_1, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\}$ are linearly dependent for every $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. By elementary linear algebra, this can only happen if $x_1, \ldots, x_{n-k}$ are linearly dependent (or if $x_1 = 0$ when $k = n-1$).

Corollary 2.2. *Any* n*-inner product space is an* $(n-k)$*-inner product space for every* $k = 1, \ldots, n-1$. *In particular, an* n*-inner product space is an inner product space.*

Corollary 2.3. *Let* $\|\cdot, \ldots, \cdot\|_n$ *be the induced* n*-norm on* X. *Then, for each* $k \in \{1, \ldots, n-1\}$, *the following function*

$$\|x_1, \ldots, x_{n-k}\| := \Bigg(\sum_{\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}} \|x_1, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\|^2 \Bigg)^{1/2},$$

defines an $(n-k)$*-norm that corresponds to* $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_{n-k}$ *on* X. *In particular,*

$$\|x\| := \Bigg(\sum_{\{i_2,\ldots,i_n\} \subseteq \{1,\ldots,n\}} \|x, a_{i_2}, \ldots, a_{i_n}\|^2 \Bigg)^{1/2},$$

defines a norm that corresponds to the derived inner product $\langle \cdot, \cdot \rangle$ *on* X.

Note that by using a derived inner product, one can develop the notion of orthogonality and the Fourier series theory in an n-inner product space, just like in an inner product space (see [3] and [13] for previous results in this direction). With respect to the derived inner product $\langle\cdot,\cdot\rangle$ defined by (1), one may observe that the set $\{a_1,\ldots,a_n\}$ is orthogonal and that $\|a_i\| = \|a_1,\ldots,a_n\|$ for every $i = 1,\ldots,n$ (see [8]). In particular, if X is n-dimensional, then $\{a_1,\ldots,a_n\}$ forms an orthogonal basis for X and each $x \in X$ can be written as $x = \|a_1,\ldots,a_n\|^{-2}\sum_{i=1}^{n}\langle x,a_i\rangle a_i$.

Unlike in [13], we can now have an orthogonal set of m vectors with $1 \le m < n$. In general, by using a derived inner product, we have a more relaxed condition for orthogonality than that in [3] or [13].

Furthermore, one may also use the derived inner products and their induced norm to study the convergence of sequences of vectors in an n-inner product space. See some recent results in [10].

2.1. Related results on n-normed spaces

Suppose now that $(X,\|\cdot,\ldots,\cdot\|_n)$ is an n-normed space and, as before, $\{a_1,\ldots,a_n\}$ is a linearly independent set in X. Then one may check that for each $k \in \{1,\ldots,n-1\}$

$$\|x_1,\ldots,x_{n-k}\| := \Bigg(\sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \|x_1,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\|^2\Bigg)^{1/2}, \tag{2}$$

defines an $(n-k)$-norm on X (see [5] and [11] for similar results). In particular, the triangle inequality can be verified as follows:

$$\begin{aligned}
&\|x+y,x_2,\ldots,x_{n-k}\| \\
&= \Bigg(\sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \|x+y,x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\|^2\Bigg)^{1/2} \\
&\le \Bigg(\sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \Big(\|x,x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\| \\
&\qquad\qquad + \|y,x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\|\Big)^2\Bigg)^{1/2} \\
&\le \Bigg(\sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \|x,x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\|^2\Bigg)^{1/2}
\end{aligned}$$

$$+\Big(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\|y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)^{1/2}$$
$$=\|x,x_2,\dots,x_{n-k}\|+\|y,x_2,\dots,x_{n-k}\|.$$

The first inequality follows from the triangle inequality for the n-norm, while the second one follows from the triangle inequality for the l^2-type norm.

In general, for $1\le p\le\infty$, one may observe that

$$\|x_1,\dots,x_{n-k}\|:=\Big(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\|x_1,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^p\Big)^{1/p},$$

also defines an $(n-k)$-norm on X. Among these derived $(n-k)$-norms, however, the case $p=2$ is special in the following sense.

Fact 2.4. *If the n-norm satisfies the parallelogram law*

$$\|x+y,x_2,\dots,x_n\|^2+\|x-y,x_2,\dots,x_n\|^2=2(\|x,x_2,\dots,x_n\|^2+\|y,x_2,\dots,x_n\|^2),$$

then the derived $(n-k)$-norm given by (2) *satisfies*

$$\|x+y,x_2,\dots,x_{n-k}\|^2+\|x-y,x_2,\dots,x_{n-k}\|^2$$
$$=2(\|x,x_2,\dots,x_{n-k}\|^2+\|y,x_2,\dots,x_{n-k}\|^2).$$

In particular, the derived norm satisfies

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2).$$

Proof. There are two ways to prove it. The first one is by establishing the parallelogram law directly. Indeed, by definition and hypothesis, we have

$$\|x+y,x_2,\dots,x_{n-k}\|^2+\|x-y,x_2,\dots,x_{n-k}\|^2$$
$$=\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\Big(\|x+y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2$$
$$+\|x-y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)$$
$$=\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}2\Big(\|x,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2+\|y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)$$
$$=2(\|x,x_2,\dots,x_{n-k}\|^2+\|y,x_2,\dots,x_{n-k}\|^2),$$

as desired.

The second way to prove it is by defining an n-inner product $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n$ on X by

$$\langle x,y|x_2,\ldots,x_n\rangle := \frac{1}{4}(\|x+y,x_2,\ldots,x_n\|^2 - \|x-y,x_2,\ldots,x_n\|^2),$$

and deriving an $(n-k)$-inner product from it with respect to $\{a_1,\ldots,a_n\}$. One will then realize that the derived $(n-k)$-norm is the induced $(n-k)$-norm from the derived $(n-k)$-inner product, and hence the parallelogram law follows.

2.2. Finite-dimensional case

Suppose here that $(X,\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n)$ is an n-inner product space of finite-dimension $d \geq n$. Then one can derive an $(n-k)$-inner product from the n-inner product in a slightly different way. To be precise, take a linearly independent set $\{a_1,\ldots,a_m\}$ in X, with $n \leq m \leq d$. With respect to $\{a_1,\ldots,a_m\}$, define for each $k \in \{1,\ldots,n-1\}$ the function $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_{n-k}$ on X^{n-k+1} by

$$\langle x,y|x_2,\ldots,x_{n-k}\rangle := \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\}} \langle x,y|x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\rangle.$$

Then we have:

Fact 2.5. *The function $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_{n-k}$ defines an $(n–k)$-inner product on X.*

Proof. Similar to the proof of Fact 2.1.

As we shall see in the next section, we may obtain an interesting inner product from the n-inner product by using a set of d, rather than just n, linearly independent vectors in X (that is, by using a basis for X).

3. Examples

We shall here present some examples showing us what sort of inner products can be derived through (1) when the n-inner product is simple, and how they are related to the original inner product.

Example 3.1. Let $X = \mathbf{R}^n$ be equipped with the standard n-inner product

$$\langle x, y|x_2, \dots, x_n\rangle := \begin{vmatrix} x\cdot y & x\cdot x_2 & \cdots & x\cdot x_n \\ x_2\cdot y & x_2\cdot x_2 & \cdots & x_2\cdot x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n\cdot y & x_n\cdot x_2 & \cdots & x_n\cdot x_n \end{vmatrix}, \tag{3}$$

where $x \cdot y$ is the usual inner product on $\mathbf{R}^n$. Then one may observe that the derived $(n-k)$-inner product with respect to an orthonormal basis $\{b_1, \dots, b_n\}$ coincides with the standard $(n-k)$-inner product on $\mathbf{R}^n$, that is,

$$\langle x, y|x_2, \dots, x_{n-k}\rangle = \begin{vmatrix} x\cdot y & x\cdot x_2 & \cdots & x\cdot x_{n-k} \\ x_2\cdot y & x_2\cdot x_2 & \cdots & x_2\cdot x_{n-k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-k}\cdot y & x_{n-k}\cdot x_2 & \cdots & x_{n-k}\cdot x_{n-k} \end{vmatrix}.$$

In particular, the derived inner product $\langle x, y\rangle$ with respect to $\{b_1, \dots, b_n\}$, which is given by

$$\langle x, y\rangle = \langle x, y|b_2, b_3, \dots, b_n\rangle + \langle x, y|b_1, b_3, \dots, b_n\rangle + \cdots + \langle x, y|b_1, b_2, \dots, b_{n-1}\rangle, \tag{4}$$

is precisely the usual inner product $x \cdot y$.

This example tells us that, on $\mathbf{R}^n$, we can define the standard n-inner product by using the usual inner product as in (3) and, conversely, derive the usual inner product from the standard n-inner product via (4).

Example 3.2. Let $X = \mathbf{R}^n$ be equipped with the standard n-inner product as in the preceeding example. Then one may verify that the derived inner product with respect to an arbitrary linearly independent set $\{a_1, \dots, a_n\}$ in X is given by

$$\langle x, y\rangle = \|a_1, \dots, a_n\|^2 \,(A^{-1}x)\cdot(A^{-1}y),$$

where A is the $n \times n$ matrix whose i-th column is the vector a_i. Note that, for every $i, j \in \{1, \dots, n\}$, we have

$$\langle a_i, a_j\rangle = \|a_1, \dots, a_n\|^2 \, b_i \cdot b_j,$$

where $\{b_1, \dots, b_n\}$ is the standard basis for $\mathbf{R}^n$. This means that $\{a_1, \dots, a_n\}$ is an orthogonal basis for $(X, \langle\cdot,\cdot\rangle)$, as remarked previously in §2.

Remark. By invoking Parseval's identity (see, e.g., [1], p. 354), Examples 3.1 and 3.2 extend to any n-dimensional inner product space X.

Example 3.3. Suppose that X is an inner product space of dimension $d \geq n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal set in X. Equip X with the standard n-inner product as in (3), with $x \cdot y$ being the inner product on X. Then one may observe that the derived inner product with respect to $\{e_1, \ldots, e_n\}$ is given by

$$\langle x, y\rangle = Px \cdot Py + n(Qx \cdot Qy),$$

where P denotes the orthogonal projection on the subspace spanned by $\{e_1, \ldots, e_n\}$ and $Q = I - P$ is its complementary projection. Notice here that its induced norm is equivalent to the original norm.

Although a little bit messy, it is also possible to obtain the expression for the derived $(n-k)$-inner product for each $k \in \{1, \ldots, n-1\}$. For example, the derived $(n-1)$-inner product with respect to $\{e_1, \ldots, e_n\}$ is given by

$$\langle x, y|x_2, \ldots, x_{n-1}\rangle$$
$$= \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_i \cdot y & x_i \cdot x_2 & \cdots & x_i \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} \cdot y & x_{n-1} \cdot x_2 & \cdots & x_{n-1} \cdot x_{n-1} \end{vmatrix} + \sum_{i=1}^{n-1} \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Qx_i \cdot Qy & Qx_i \cdot Qx_2 & \cdots & Qx_i \cdot Qx_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} \cdot y & x_{n-1} \cdot x_2 & \cdots & x_{n-1} \cdot x_{n-1} \end{vmatrix},$$

with x_1 being identified as x.

Example 3.4. Let $X = \mathbf{R}^d$ be equipped with the standard n-inner product as in (3), with $x \cdot y$ being the usual inner product on $\mathbf{R}^d$. Then one may particularly observe that the derived inner product with respect to an orthonormal basis $\{b_1, \ldots, b_d\}$ is given by

$$\langle x, y\rangle = \sum_{\{i_2,\ldots,i_n\}\subseteq\{1,\ldots,d\}} \langle x, y|b_{i_2}, \ldots, b_{i_n}\rangle = C_{n-1}^{d-1}\, x \cdot y,$$

where $C_{n-1}^{d-1} = \frac{(d-1)!}{(d-n)!(n-1)!}$. This derived inner product is better than the previous one in the sense that it is only a multiple of the usual inner product.

Remark. By invoking Parseval's identity, Example 3.4 may also be extended to any finite d-dimensional inner product space X.

Acknowledgment

This work was carried out during the author's visit to the School of Mathematics, UNSW, Sydney, in 2000/2001. The author was sponsored by an Australia-Indonesia Merdeka Fellowship funded by the Australian Government through the Department of Education, Training and Youth Affairs and promoted through Australia Education International. The author thanks the referees for their useful suggestions.

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